# On a generalized Connes-Hochschild-Kostant-Rosenberg theorem

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**Abstract**

The central result of this paper is an explicit computation of the Hochschild and cyclic homologies of a natural smooth subalgebra of stable continuous trace algebras having smooth manifolds $X$ as their spectrum. More precisely, the Hochschild homology is identified with the space of differential forms on $X$, and the periodic cyclic homology with the twisted de Rham cohomology of $X$, thereby generalizing some fundamental results of Connes and Hochschild-Kostant-Rosenberg. The Connes-Chern character is also identified here with the twisted Chern character.

*Key words:* Cyclic homology, Hochschild homology, K-theory, continuous trace $C^*$-algebras, smooth algebras, Dixmier-Douady invariant, twisted K-theory, twisted cohomology, Connes-Chern character, twisted Chern character.
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## 1 Introduction

In [11], Connes showed how to extend both classical Hochschild homology and cyclic homology to the category of locally convex algebras, computing in particular the Hochschild homology of the Fréchet algebra of smooth functions on a compact manifold $X$. This he showed was canonically isomorphic to the space of differential forms on $X$, thus generalizing a fundamental result of Hochschild-Kostant-Rosenberg [24]. In the same paper, Connes also identified

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the periodic cyclic homology of the Fréchet algebra of smooth functions on $X$ with the de Rham cohomology of $X$. Connes's theorem shows that cyclic homology is indeed a far-reaching generalisation of de Rham cohomology for noncommutative topological algebras, and serves as a central motivation for the results here.

Since [11], there have been extensions in different directions including the case of non-compact manifolds due to Pflaum [31], the equivariant case studied by Block and Getzler [4] and the case of étale groupoids and foliations [9,12]. In this paper we take up the suggestion of [7] and prove the following generalisation of the Connes and Hochschild-Kostant-Rosenberg theorem. Let $A$ be a stable continuous trace $C^*$-algebra with spectrum a smooth, compact manifold $X$. Then by [17], $A = C(X, \mathcal{K}(P))$ is the algebra of continuous sections of a locally trivial bundle $\mathcal{K}(P) = P \times_{PU} \mathcal{K}$ on $X$ with fibre the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space associated to a principal $PU$ bundle $P$ on $X$ via the adjoint action of $PU$ on $\mathcal{K}$. Here $PU$ is the group of projective unitary operators on the Hilbert space. Such algebras $A$ are classified up to isomorphism by their Dixmier-Douady invariant $\delta(P) \in H^3(X;\mathbb{Z})$. Standard approximation theorems show that the transition functions of $P$ can be taken to be smooth and so inside $A$ we can consider a dense, canonical smooth $*$-subalgebra $\mathcal{A} = C^\infty(X, \mathcal{L}^1(P))$ consisting of all smooth sections of the sub-bundle $\mathcal{L}^1(P) = P \times_{PU} \mathcal{L}^1$ of $\mathcal{K}(P)$ with fibre the algebra $\mathcal{L}^1$ of trace class operators on the Hilbert space and structure group $PU$. Then a central result in this paper is the following,

**Theorem 1.1** *The Hochschild homology $HH_\bullet(\mathcal{A})$ of the Fréchet algebra $\mathcal{A}$ is isomorphic to the space of differential forms $\Omega^\bullet(X)$ and the periodic cyclic homology $HP_\bullet(\mathcal{A})$ is isomorphic to the twisted de Rham cohomology $H^\bullet(X; c(P))$ for some closed 3-form $c(P)$ on $X$ such that $\frac{1}{2\pi i}c(P)$ represents the image of $\delta(P)$ in real cohomology.*

Recall from [6] that the twisted de Rham cohomology $H^\bullet(X; c(P))$ of the manifold $X$ is defined as the cohomology of the complex $(\Omega^\bullet(X), \mathrm{d} - c)$, where $c$ denotes exterior multiplication by the closed 3-form $c = c(P)$. By choosing a connection $\nabla$ on $\mathcal{L}^1(P)$ satisfying a derivation property with respect to the algebra structure we construct a chain map Ch from the complex computing periodic cyclic homology to the twisted de Rham complex $(\Omega^\bullet(X), \mathrm{d} - c)$. By examining the behaviour of this map when we perturb the connection $\nabla$ we are able to deduce that this chain map is locally a quasi-isomorphism on $X$. Standard double complex arguments then give that Ch is a quasi-isomorphism globally. Extensions to $*$-algebras of smooth sections of a smooth algebra bundle with fibre the algebra of operators belonging to the Schatten class ideal $\mathcal{L}^p$ are also discussed.

Recall from [34] that the twisted $K$-theory is by definition $K^i(X, P) = K_i(A)$,

which is in turn isomorphic to $K_i(\mathcal{A})$ as argued in section 4.2. In [6,27], it was established that the twisted $K$-theory $K_\bullet(\mathcal{A})$ could be identified with so-called 'bundle gerbe $K$-theory', $K^\bullet_{bg}(X,P)$ (also denoted by $K^\bullet(X,P)$), thus giving a geometric description of elements of twisted $K$-theory. Using this geometric description, it was established that there is a twisted Chern character $\mathrm{ch}_P : K^\bullet(X,P) \to H^\bullet(X,c(P))$ possessing certain functorial properties.

It turns out that the Connes-Chern character homomorphism $\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$ satisfies the same functorial properties. This is not surprising, once we establish that under the natural identifications $K_\bullet(\mathcal{A})$ with $K^\bullet(X,P)$, and $HP_\bullet(\mathcal{A})$ with $H^\bullet(X,c(P))$, the Connes-Chern character homomorphism agrees with the twisted Chern character homomorphism constructed in [6], which can be encapsulated as the commutativity of the following diagram,

$$\begin{array}{ccc} K^\bullet(X,P) & \xrightarrow{\cong} & K_\bullet(\mathcal{A}) \\ {\scriptstyle \mathrm{ch}_P}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ H^\bullet(X,c(P)) & \xrightarrow{\cong} & HP_\bullet(\mathcal{A}). \end{array} \tag{1}$$

We now outline the paper. In section 2 we first give some standard background material on topological algebras and establish a result (Proposition 2.3) that we will need later in the paper. In section 3 we review preliminary material on defining the Hochschild, cyclic and periodic cyclic homology groups $HH_\bullet(\mathcal{A})$, $HC(\mathcal{A})$ and $HP_\bullet(\mathcal{A})$ respectively for a Fréchet algebra $\mathcal{A}$, and the computation of some examples, such as $HH_\bullet(\mathcal{L}^1), HC_\bullet(\mathcal{L}^1)$ and $HP(\mathcal{L}^1)$. In Section 4, we review the $K$-theory of such algebras, and specialize to the case of interest for the rest of the paper, namely $\mathcal{A} = C^\infty(X,\mathcal{L}^1(P))$, where $P$ is a principal $PU$-bundle with Dixmier-Douady class equal to $[c] \in H^3(X,\mathbb{Z})$, and $\mathcal{L}^1(P) = P \times_{PU} \mathcal{L}^1$ is the bundle with fibre $\mathcal{L}^1$ associated to $P$ via the adjoint action of $PU$ on $\mathcal{L}^1$. We show that the $K$-theory $K_\bullet(\mathcal{A})$ can be identified with the twisted $K$-theory $K^\bullet(X;P)$. We also review the construction of the Connes-Chern character $\mathrm{ch} : K_\bullet(\mathcal{A}) \to HP_\bullet(\mathcal{A})$. In Section 5 we construct, following ideas of Gorokhovsky [21,22], a chain map $\mathrm{Ch}_\bullet\colon CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u)) \to \Omega^\bullet(X)\otimes\mathbb{C}((u))$ from the complex $CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u))$ computing periodic cyclic homology to the twisted de Rham complex $(\Omega^\bullet(X)\otimes\mathbb{C}((u)), \mathrm{d}+uc)$. We remark that the map Ch can be constructed in the formalism of Quillen [32]. In Theorem 5.9 we establish that the map $\mathrm{Ch}_\bullet$ is a quasi-isomorphism. We also describe extensions of this result to Hochschild and cyclic homology respectively. The arguments we use here are reminiscent of the Čech-de Rham tic-tac-toe type argument. Finally in Section 6, we prove the commutativity of the diagram (1). We also prove that the Connes-Chern map $\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$ becomes an isomorphism on tensoring with the complex numbers.

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## 2 Preliminaries on Topological Algebras

A *Fréchet algebra* $A$ is an algebra with a complete locally convex topology determined by a countable family of semi-norms $\{p_\alpha\}$ which are required to be sub-multiplicative in the sense that $p_\alpha(ab) \leq p_\alpha(a)p_\alpha(b)$. This implies that the multiplication map $A \times A \to A$ is jointly continuous. We could in fact consider a larger class of algebras, so-called $m$-algebras for which we refer to Cuntz's article [13], but for the purposes of this note Fréchet algebras will suffice.

Let us also remark that the unitization $\tilde{A}$ of a Fréchet algebra $A$ is again a Fréchet algebra. Recall that $\tilde{A} = A \oplus \mathbb{C}$ with multiplication given by $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. The topology of $\tilde{A}$ is determined by the family of semi-norms $\{\tilde{p}_\alpha\}$ given by $\tilde{p}_\alpha(a, \lambda) = p_\alpha(a) + |\lambda|$. Similarly the direct sum $A \oplus B$ of two Fréchet algebras is again Fréchet. One needs to take more care in defining tensor products, the tensor product in which we will be exclusively interested in is Grothendieck's projective tensor product $\otimes_\pi$ [36]. Recall that this is defined as follows: if $A$ and $B$ are Fréchet algebras with countable families of semi-norms $\{p_\alpha\}$ and $\{q_\beta\}$ then the algebraic tensor product $A \odot B$ is given a locally convex topology, called the *projective tensor product* topology, defined by the countable family of semi-norms given by

$$p_\alpha \otimes q_\beta(c) = \inf\{\sum_{i=1}^{n} p_\alpha(a_i) q_\beta(b_i) |\ c = \sum_{i=1}^{n} a_i \odot b_i\}$$

We will denote the algebraic tensor product $A \odot B$ equipped with this topology by $A \otimes_\pi B$ and the completion of $A \otimes_\pi B$ by $A \hat{\otimes}_\pi B$. $A \hat{\otimes}_\pi B$ is again a Fréchet algebra.

Recall the well known example of the Fréchet algebra $C^\infty(X)$ of smooth functions on a Riemannian manifold $X$. The seminorms on this algebra are defined as follows. Let $V$ be a chart in $X$ and $\{K_n\}_{n \in \mathbb{N}}$ be an exhausting sequence of compact subsets of $V$, i.e., $K_n$ is contained in the interior of $K_{n+1}$ and every compact subset of $V$ is contained in some $K_n$. For $f \in C^\infty(X)$, we denote its restriction to $V$ (and to $K_n$) by the same symbol. Then set

$$p_{n,V}(f) = \sum_{j=0}^{n} \frac{1}{j!} \sup_{x \in K_n} |\partial^\alpha f(x)|.$$

where the partial derivatives are taken in the chart $V$. Then varying over all charts $V$ and all $n \in \mathbb{N}$, $p_{n,V}$ forms a basis of continuous seminorms for $C^\infty(X)$, and the derivation property of the partial derivatives ensures that the family of semi-norms is sub-multiplicative, giving it the structure of a Fréchet algebra.

Any Banach algebra $\mathcal{A}$ is a Fréchet algebra with the family of semi-norms consisting of the single norm $|\cdot|$. In particular we will be interested in the case where $\mathcal{A} = \mathcal{L}^1(H)$, the Banach algebra of trace class operators on a separable Hilbert space $H$. We take this opportunity to review some facts about $\mathcal{L}^1(H)$. Let $H$ be an infinite dimensional Hilbert space and $\{v_n\},\ n \in \mathbb{N}$ an orthonormal basis of vectors in $H$. Let $\mathcal{B} = \mathcal{B}(H)$ denote the bounded operators on $H$. We can define a semi-finite trace on positive elements in $\mathcal{B}$ by

$$\mathrm{tr}\,(x) = \sum_{n=1}^{\infty} (x, v_n). \tag{2}$$

It turns out that $\mathrm{tr}\,(x)$ is well defined, independent of the choice of orthonormal basis and satisfies $\mathrm{tr}\,(x^*x) = \mathrm{tr}\,(xx^*)$ for all $x \in \mathcal{B}$. For $p \geq 1$, define the Schatten class

$$\mathcal{L}^p = \{x \in \mathcal{B} : \mathrm{tr}\,(|x|^p) < \infty\}\,.$$

where $|x| = \sqrt{x^*x}$. Then it is well known that $\mathcal{L}^p$ is a ideal in $\mathcal{B}$ that is closed under taking adjoints, and which is contained in the closed ideal of compact operators $\mathcal{K}$ in $\mathcal{B}$. $\mathcal{L}^p$ is a Banach $*$-algebra with respect to the norm $||x||_p = (\mathrm{tr}\,(|x|^p))^{1/p}$, but it has no unit. $\mathcal{L}^1$ is the Banach $*$-algebra of trace class operators and $\mathcal{L}^2$ the Banach $*$-algebra of Hilbert-Schmidt operators. Moreover, it is the case that the trace map is well defined for arbitrary elements of $\mathcal{L}^1$,

$$\mathrm{tr}\,: \mathcal{L}^1 \to \mathbb{C} \tag{3}$$

and is given by the same formula (2). Moreover $\mathcal{L}^p \subset \mathcal{L}^q$ whenever $p \leq q$.

**Lemma 2.1** *Suppose that $X$ and $Y$ are paracompact manifolds. Then one has a canonical isomorphism,*

$$C^\infty(X, \mathcal{L}^1)\hat{\otimes}_\pi C^\infty(Y, \mathcal{L}^1) \cong C^\infty(X \times Y, \mathcal{L}^1 \hat{\otimes}_\pi \mathcal{L}^1).$$

**Proof.** Recall that the projective tensor product satisfies the following two properties (see for example [36]). If $X$ and $Y$ are smooth manifolds then $C^\infty(X)\hat{\otimes}_\pi C^\infty(Y) = C^\infty(X \times Y)$ and if $E$ is a Fréchet space then $C^\infty(X)\hat{\otimes}_\pi E = C^\infty(X, E)$, which is the Fréchet space of $E$-valued smooth functions on $X$.

Therefore $C^\infty(X,\mathcal{L}^1) \cong C^\infty(X)\hat{\otimes}_\pi\mathcal{L}^1$ and $C^\infty(Y,\mathcal{L}^1) \cong C^\infty(Y)\hat{\otimes}_\pi\mathcal{L}^1$, and

$$\begin{aligned} C^\infty(X,\mathcal{L}^1)\hat{\otimes}_\pi C^\infty(Y,\mathcal{L}^1) &\cong C^\infty(X)\hat{\otimes}_\pi C^\infty(Y)\hat{\otimes}_\pi\mathcal{L}^1\hat{\otimes}_\pi\mathcal{L}^1 \\ &\cong C^\infty(X\times Y)\hat{\otimes}_\pi\mathcal{L}^1\hat{\otimes}_\pi\mathcal{L}^1 \\ &\cong C^\infty(X\times Y,\mathcal{L}^1\,\hat{\otimes}_\pi\,\mathcal{L}^1). \end{aligned}$$

**Remark 2.2** *We caution the reader that the projective tensor product, $\mathcal{L}^1\,\hat{\otimes}_\pi\,\mathcal{L}^1$ is not isomorphic to $\mathcal{L}^1$. A similar remark applies when $\mathcal{L}^1$ is replaced by $\mathcal{L}^p$ or by $\mathcal{K}$. This is in contrast to the well known fact that $\mathcal{K}\otimes\mathcal{K}\cong\mathcal{K}$, when the $C^*$ tensor product is applied instead.*

Another Fréchet algebra that we will be interested in is $C^\infty(X,\mathcal{L}^1(P))$, the algebra of smooth sections of a Banach algebra bundle $\mathcal{L}^1(P) = P\times_{PU}\mathcal{L}^1$ with fibre the algebra of trace class operators $\mathcal{L}^1$ associated to a principal $PU$ bundle $P$ via the adjoint action of $PU$ on $\mathcal{L}^1$. Here $PU = PU(H)$ is the projective unitary group of $H$ equipped with the norm topology (see §4.2 for more details). The trace $\mathrm{tr}\,(\cdot)\colon \mathcal{L}^1(H)\to\mathbb{C}$ can be extended fibrewise $\mathrm{tr}\,(\cdot)\colon C^\infty(\mathcal{L}^1(P))\to C^\infty(X)$, and the algebra of sections $C^\infty(\mathcal{L}^1(P)) = C^\infty(X,\mathcal{L}^1(P))$ is made into a Fréchet algebra as follows. Let $V$ be a chart in $X$ and $\{K_n\}_{n\in\mathbb{N}}$ be an exhausting sequence of compact subsets of $V$. For $f\in C^\infty(X,\mathcal{L}^1(P))$, we denote its restriction to $V$ (and to $K_n$) by the same symbol. Then set

$$p_{n,V}(f) = \sum_{j=0}^{n}\frac{1}{j!}\sup_{x\in K_n}\mathrm{tr}\,(|\nabla^j f(x)|)$$

where $\nabla^j f$ is the $j$-th covariant derivative of the section $f$ taken the the chart $V$, $\nabla$ is a connection on the bundle $\mathcal{L}^1(P)$ which acts as a derivation on $C^\infty(X,\mathcal{L}^1(P))$ in the sense that $\nabla(fh) = f\nabla h + \nabla f h$ (the existence of such connections will be established in §5.1.). Then varying over all charts $V$ and all $n\in\mathbb{N}$, $p_{n,V}$ forms a basis of continuous seminorms for $C^\infty(X)$, giving it the structure of a Fréchet algebra. If $\nabla'$ is another connection on $\mathcal{L}^1(P)$ that is also compatible with the product as above, then $\nabla'$ defines a quasi-isometric family of semi-norms $\{p'_{n,V}\}$ provided that $\nabla$ and $\nabla'$ differ by a 1-form on $X$ which takes values in the sub-bundle $\mathcal{B}(\mathcal{L}^1(P))\subseteq\mathrm{End}(\mathcal{L}^1(P))$ of *bounded* endomorphisms of $\mathcal{L}^1(P)$. The derivation property of $\nabla$ ensures that the family of semi-norms is sub-multiplicative.

The following is a generalization of Lemma 2.1 to the case of non-trivial bundles of algebras with fibre $\mathcal{L}^1$ and structure group $PU$.

**Proposition 2.3** *Suppose that $P$ and $Q$ are principal $PU$ bundles on paracompact manifolds $X$ and $Y$ respectively. Denote by $\mathcal{L}^1(P)$ and $\mathcal{L}^1(Q)$ the Banach vector bundles on $X$ and $Y$ respectively with fibre $\mathcal{L}^1$ associated to $P$ and $Q$ via the adjoint action of $PU$ on $\mathcal{L}^1$. Suppose also $\mathcal{L}^1(P)$ and $\mathcal{L}^1(Q)$ come*

*equipped with connections $\nabla_{\mathcal{L}^1(P)}$ and $\nabla_{\mathcal{L}^1(Q)}$ satisfying the derivation property above. If $C^\infty(X, \mathcal{L}^1(P))$ and $C^\infty(Y, \mathcal{L}^1(Q))$ are equipped with the topologies coming from the families of semi-norms induced by $\nabla_{\mathcal{L}^1(P)}$ and $\nabla_{\mathcal{L}^1(Q)}$, then we have*

$$C^\infty(X, \mathcal{L}^1(P)) \hat{\otimes}_\pi C^\infty(Y, \mathcal{L}^1(Q)) = C^\infty(X \times Y, \mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q)),$$

*where $\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q)$ is the algebra bundle over the cartesian product $X \times Y$, with fibre $(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))_{(x,y)} = \mathcal{L}^1(P)_x \hat{\otimes}_\pi \mathcal{L}^1(Q)_y$, i.e. modelled on $\mathcal{L}^1 \hat{\otimes}_\pi \mathcal{L}^1$.*

**Proof.** We note first that $C^\infty(\mathcal{L}^1(P)) \odot C^\infty(\mathcal{L}^1(Q))$ can be identified with a subspace of $C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$. We will prove two things:

(1) $C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(Q)) \subset C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$ is a homeomorphism onto its image,
(2) $C^\infty(\mathcal{L}^1(P)) \odot C^\infty(\mathcal{L}^1(Q))$ is dense in $C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$.

This suffices to show the equality $C^\infty(\mathcal{L}^1(P)) \hat{\otimes}_\pi C^\infty(\mathcal{L}^1(Q)) = C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$. To show that $C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(Q)) \subset C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$ is a homeomorphism onto its image, first of all observe that the inclusion $C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(Q)) \subset C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$ is continuous. Now suppose we have a sequence $f_i$ in $C^\infty(\mathcal{L}^1(P)) \odot C^\infty(\mathcal{L}^1(Q))$ which converges to zero in the $C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q))$ topology. We have to show that $f_i \to 0$ in the projective tensor product topology on $C^\infty(\mathcal{L}^1(P)) \odot C^\infty(\mathcal{L}^1(Q))$. Choose good open covers $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$ of $X$ and $Y$ respectively and denote by $\mathcal{L}^1(P)_\alpha$ the restriction $\mathcal{L}^1(P)|_{U_\alpha}$ and by $\mathcal{L}^1(Q)_\beta$ the restriction of $\mathcal{L}^1(Q)|_{V_\beta}$. Since $U_\alpha, V_\beta$ are contractible, the restricted bundles $\mathcal{L}^1(P)_\alpha, \mathcal{L}^1(Q)_\beta$ are trivializable, therefore $C^\infty(\mathcal{L}^1(P)_\alpha) \cong C^\infty(U_\alpha, \mathcal{L}^1)$ and $C^\infty(\mathcal{L}^1(Q)_\beta) \cong C^\infty(V_\beta, \mathcal{L}^1)$. By remarks above and by Lemma 2.1, we have the canonical isomorphism

$$C^\infty(\mathcal{L}^1(P)_\alpha) \hat{\otimes}_\pi C^\infty(\mathcal{L}^1(Q)_\beta) = C^\infty(\mathcal{L}^1(P)_\alpha \boxtimes \mathcal{L}^1(Q)_\beta).$$

Denote by $r_\alpha \colon C^\infty(\mathcal{L}^1(P)) \to C^\infty(\mathcal{L}^1(P)_\alpha)$, $r_\beta \colon C^\infty(\mathcal{L}^1(Q)) \to C^\infty(\mathcal{L}^1(Q)_\beta)$ and $r_{\alpha,\beta} \colon C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q)) \to C^\infty(\mathcal{L}^1(P)_\alpha \boxtimes \mathcal{L}^1(Q)_\beta)$ the restriction maps. Then we have a commutative diagram

$$\begin{tikzcd}
C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(Q)) \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{inc}"'] & C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q)) \arrow[d, "||"] \\
C^\infty(\mathcal{L}^1(P)) \hat{\otimes}_\pi C^\infty(\mathcal{L}^1(Q)) \arrow[d, "\prod (r_\alpha \hat{\otimes}_\pi r_\beta)"'] & C^\infty(\mathcal{L}^1(P) \boxtimes \mathcal{L}^1(Q)) \arrow[d, "r_{\alpha,\beta}"] \\
\prod_{\alpha,\beta} C^\infty(\mathcal{L}^1(P)_\alpha) \hat{\otimes}_\pi C^\infty(\mathcal{L}^1(Q)_\beta) \arrow[r, "="] & \prod_{\alpha,\beta} C^\infty(\mathcal{L}^1(P)_\alpha \boxtimes \mathcal{L}^1(Q)_\beta)
\end{tikzcd} \tag{4}$$

where $\prod$ denotes the direct product. Both the lower vertical maps are split injective (a splitting can be constructed via partitions of unity subordinate to the open covers $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$). It follows from the commutativity of the diagram (4) that $r_\alpha\hat{\otimes}_\pi r_\beta(f_i)\to 0$ and hence $f_i\to 0$ since $\prod r_\alpha\hat{\otimes}_\pi r_\beta$ is split injective. Therefore the inclusion $C^\infty(\mathcal{L}^1(P))\otimes_\pi C^\infty(\mathcal{L}^1(Q))\subset C^\infty(\mathcal{L}^1(P)\boxtimes\mathcal{L}^1(Q))$ is a homeomorphism onto its image. We now have to show that $C^\infty(\mathcal{L}^1(P))\odot C^\infty(\mathcal{L}^1(Q))$ is dense in $C^\infty(\mathcal{L}^1(P)\boxtimes\mathcal{L}^1(Q))$. Let $f\in C^\infty(\mathcal{L}^1(P)\boxtimes\mathcal{L}^1(Q))$ and let $f_{\alpha,\beta}=r_{\alpha,\beta}(f)$ denote its image in $C^\infty(\mathcal{L}^1(P)_\alpha\boxtimes\mathcal{L}^1(Q)_\beta)=C^\infty(\mathcal{L}^1(P)_\alpha)\hat{\otimes}_\pi C^\infty(\mathcal{L}^1(Q)_\beta)$ under the restriction map $r_{\alpha,\beta}$. Choose a sequence $f^i_{\alpha,\beta}\in C^\infty(\text{Ł}^1(P)_\alpha)\odot C^\infty(\mathcal{L}^1(Q)_\beta)$ converging to $f_{\alpha,\beta}$. Do this for all $\alpha$ and $\beta$ and denote by $f_i$ the image of the sequence $\prod f^i_{\alpha,\beta}$ under the splitting for the map $\prod r_\alpha\odot r_\beta$. It follows from the commutativity of the diagram (4) above that $f_i$ converges to $f$ in $C^\infty(\mathcal{L}^1(P)\boxtimes\mathcal{L}^1(Q))$.

## 3 Cyclic Homology

### *3.1 Definitions*

In this section we recall the definition and main properties of Hochschild, cyclic and periodic cyclic homology. We shall closely follow the articles of Block and Getzler [4] and Cuntz [13]. Let $A$ be a unital algebra over the complex numbers $\mathbb{C}$ and let $CC_k(A)=A^{\otimes k+1}$. Define a differential $b$ of degree $-1$ by

$$\begin{aligned}b(a_0\otimes a_1\otimes\cdots\otimes a_k)=&\\ &a_0a_1\otimes a_2\otimes\cdots\otimes a_k+\sum_{j=2}^{n}(-1)^{j-1}a_0\otimes\cdots\otimes a_{j-1}a_j\otimes\cdots\otimes a_k\\ &\qquad\qquad+(-1)^n a_na_0\otimes a_1\otimes\cdots\otimes a_{k-1}. \qquad (5)\end{aligned}$$

$b$ is defined to be zero on $CC_0$. The *Hochschild homology* $HH_\bullet(A)$ is the homology of the complex $(CC_\bullet(A),b)$. This definition extends to the non-unital case as described in Loday [25] pages 29–30 by setting

$$HH_n(A)=\operatorname{coker}\Big(HH_n(\mathbb{C})\to HH_n(\tilde{A})\Big)$$

for a non-unital algebra $A$, where $\tilde{A}$ denotes the unitization of $A$. If however, $A$ is *$H$-unital*, then $HH_n(A)$ can be computed from the same complex $(CC_\bullet(A),b)$ which computes Hochschild homology in the unital case. Recall [25] that $A$ is said to be $H$-unital if the bar complex $\mathrm{Bar}_\bullet(A)$ is contractible. The bar complex has $\mathrm{Bar}_k(A)=A^{\otimes k}$ in degree $k$ with differential $b'$ given by

the standard formula

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}) = \sum_{j=0}^{k-2} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{k-1}.$$

Another way to compute $HH_n(A)$, which works regardless of whether $A$ is unital or not, is to use the *reduced* Hochschild complex which is defined to be $\overline{CC}_k(A) = \tilde{A} \otimes A^{\otimes k} = A^{\otimes k} \oplus A^{\otimes k+1}$ in degrees $k > 0$ and $\overline{CC}_0(A) = A$ with differential $b$ given by the obvious extension of the formula (5) above. More specifically, the operator $b\colon \overline{CC}_{k+1}(A) \to \overline{CC}_k(A)$ can be written [13] with respect to the splitting $\overline{CC}_{k+1}(A) = A^{\otimes k+1} \oplus A^{\otimes k+2}$ as

$$b = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix},$$

where $\lambda\colon A^{\otimes k} \to A^{\otimes k}$ is Connes' signed permutation operator $\lambda(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}) = (-1)^k a_{k-1} \otimes a_0 \otimes \cdots \otimes a_{k-2}$. From now on we will write $CC_k(A)$ for $\overline{CC}_k(A)$. We also introduce Connes' operator $B$ of degree 1 defined by

$$B(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

$b$ and $B$ satisfy the identities $b^2 = B^2 = bB + Bb = 0$. We can also write $B$, with respect to the splitting $CC_k(A) = A^{\otimes k+1} \oplus A^{\otimes k}$, as

$$B = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$$

where $Q = \sum_{j=0}^{k} \lambda^k$. Following Block and Getzler, we introduce a variable $u$ of degree $-2$. We then define the *cyclic homology* $HC_\bullet(A)$ of $A$ to be the homology of the complex

$$(CC_\bullet(A) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]], b + uB)$$

while the *periodic cyclic homology* $HP_\bullet(A)$ of $A$ is the homology of the complex

$$(CC_\bullet(A) \otimes \mathbb{C}((u)), b + uB).$$

Note that the periodic theory $HP_\bullet(A)$ is a $\mathbb{Z}$-graded version of the usual $\mathbb{Z}/2$-graded version. In this note we will be concerned with Hochschild, cyclic and periodic cyclic homology defined for *topological* algebras. Following [11] we shall define *topological* Hochschild, cyclic and periodic cyclic homology for Fréchet algebras $\mathcal{A}$. Suppose that $\mathcal{A}$ is a Fréchet algebra. Then we form the vector spaces $CC_k(\mathcal{A}) = \tilde{\mathcal{A}} \hat{\otimes}_\pi \mathcal{A}^{\hat{\otimes}_\pi k}$ by replacing the ordinary tensor products $\otimes$ with the completed projective tensor product $\hat{\otimes}_\pi$ of Grothendieck. The

operators $b$ and $B$ are all continuous and extend to the completed topological vector spaces. We then define $HH_\bullet(\mathcal{A})$, $HC_\bullet(\mathcal{A})$ and $HP_\bullet(\mathcal{A})$ for a Fréchet algebra in exactly the same manner as above using the complexes formed by replacing ordinary tensor products with projective tensor products. Clearly the groups $HH_\bullet(\mathcal{A})$, $HC_\bullet(\mathcal{A})$ and $HP_\bullet(\mathcal{A})$ are all functorial with respect to continuous homomorphisms $\phi\colon \mathcal{A} \to \mathcal{B}$ of Fréchet algebras. It is also possible to define a (topological) *bivariant* version $HP_\bullet(\mathcal{A}, \mathcal{B})$ of periodic cyclic theory [16,13,14] however we shall make no use of this.

Finally, we summarise some key properties of periodic cyclic homology, thought of as a functor $HP_\bullet$ from some suitable class of locally convex algebras to the category of graded abelian groups (we refer to [16,13,14] for more details)

($HP1$): $HP_\bullet$ is *diffeotopy invariant*, i.e. if $\alpha_0, \alpha_1\colon \mathcal{A} \to \mathcal{B}$ are differentiably homotopic (so that there exists a continuous homomorphism $\alpha\colon \mathcal{A} \to C^\infty([0,1])\hat{\otimes}_\pi\mathcal{B}$ with $\alpha(a)(0) = \alpha_0(a)$ and $\alpha(a)(1) = \alpha_1(a)$ for all $a \in \mathcal{A}$) then the induced homomorphisms $(\alpha_0)_*, (\alpha_1)_*\colon HP_\bullet(\mathcal{A}) \to HP_\bullet(\mathcal{B})$ are equal.

($HP2$): $HP_\bullet$ is *stable*, i.e. if $\mathcal{L}^1$ denotes the trace class operators on a separable Hilbert space $H$, then $HP_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1)$ is isomorphic to $HP_\bullet(\mathcal{A})$.

($HP3$): $HP_\bullet$ satisfies *excision*, i.e. if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of algebras which admits a continuous linear splitting, then we have the six-term exact sequence

$$\begin{array}{ccccc} HP_0(\mathcal{A}) & \to & HP_0(\mathcal{B}) & \to & HP_0(\mathcal{C}) \\ \uparrow\delta & & & & \downarrow\delta \\ HP_1(\mathcal{C}) & \leftarrow & HP_1(\mathcal{B}) & \leftarrow & HP_1(\mathcal{A}) \end{array}$$

Of these, property ($HP2$) will be particularly important; we discuss it in more detail in § 3.3 below.

### *3.2 The Cyclic Homology of Smooth Functions on a Manifold*

We will be interested in the Fréchet algebra $\mathcal{A} = C^\infty(X)$ of smooth functions on a manifold $X$. Here the answer was first computed by Connes in [11].

**Proposition 3.1 ([11,31])** *Let $\mathcal{A} = C^\infty(X)$ be the Fréchet algebra of smooth functions on a paracompact manifold $X$. Then there is an isomorphism between the (continuous) Hochschild homology groups $HH_n(\mathcal{A})$ of $\mathcal{A}$ and the differential $n$-forms $\Omega^n(X)$ on $X$. The cyclic homology groups $HC_n(\mathcal{A})$ can be*

*identified with*

$$HC_n(\mathcal{A}) = \Omega^n(X)/\mathrm{d}\Omega^{n-1}(X) \oplus uH^{n-2}_{dR}(X) \oplus u^2 H^{n-4}_{dR}(X) \oplus \cdots$$

*which terminates in* $u^{\frac{n}{2}}H^0_{dR}(X)$ *if* $n$ *is even and in* $u^{\frac{n-1}{2}}H^1_{dR}(X)$ *if* $n$ *is odd. Finally* $HP_n(\mathcal{A}) = H^{\mathrm{ev}}_{dR}(X)u^n$ *if* $n$ *is even and* $HP_n(\mathcal{A}) = H^{\mathrm{odd}}_{dR}(X)u^n$ *if* $n$ *is odd.*

In [11] Connes proved this result under the assumption that $X$ was compact. Connes showed that a Kozul complex gave a topological projective resolution of $\mathcal{A}$ and was able to identify the Hochschild cohomology of $\mathcal{A}$ with the space of de Rham currents on $X$. Pflaum [31] later removed the restriction that $X$ be compact. The map $\phi_k\colon \mathcal{A}^{\otimes k+1} \to \Omega^k(X)$ defined by

$$\phi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \frac{1}{k!}a_0\mathrm{d}a_1 \cdots \mathrm{d}a_k \tag{6}$$

gives a map of complexes $\phi_k$ from $CC_k(\mathcal{A}) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$ to $\Omega^k(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$ where the latter complex is equipped with the differential $ud$. Note that $\phi_k$ extends to the completion of $CC_k(\mathcal{A})$ in the projective tensor product topology. Similarly, $\phi_k$ induces a map of complexes $\phi_k\colon CC_k(\mathcal{A}) \otimes \mathbb{C}((u)) \to \Omega^k(X) \otimes \mathbb{C}((u))$. It can be shown that the map $\phi_k$ induces an isomorphism on the homology of these complexes.

### *3.3 Topologically H-unital algebras and topological Morita invariance*

In this section we describe the cyclic homology of the Schatten ideals. We begin with the following Lemma from the thesis [2] of Baehr — for which we are very grateful to Joachim Cuntz for making available to us.

**Lemma 3.2** *Suppose* $\mathcal{A}$ *is a Fréchet algebra such that the multiplication map* $m\colon \mathcal{A}\hat{\otimes}_\pi\mathcal{A} \to \mathcal{A}$ *has a continuous left* $\mathcal{A}$*-linear section* $s\colon \mathcal{A} \to \mathcal{A}\hat{\otimes}_\pi\mathcal{A}$*. Then* $\mathcal{A}$ *is topologically* $H$*-unital.*

The point is the section $s$ allows one to define a contracting homotopy of the (topological) bar complex of $\mathcal{A}$. In particular (following [2]), this allows us to prove that $\mathcal{L}^1$ is topologically $H$-unital. To see this, recall that there is an identification $\mathcal{L}^1 = H\hat{\otimes}_\pi H$ under which the multiplication map corresponds to the continuous map defined by $a \otimes b \mapsto a(u) \otimes b^*(u)$ for a fixed unit vector $u \in H$. We define a continuous left $\mathcal{L}^1$-linear section of the multiplication map as follows:

$$\begin{aligned} s : \mathcal{L}^1 = H\hat{\otimes}_\pi H &\to \mathcal{L}^1 \otimes \mathcal{L}^1 \\ v \otimes w &\to \langle \cdot, u\rangle v \otimes \langle \cdot, w\rangle u \end{aligned} \tag{7}$$

where $\langle\, ,\, \rangle$ denotes the inner product on $H$. Therefore, one concludes that $\mathcal{L}^1$ is topologically $H$-unital. We make the remark, following [2], that $\mathcal{L}^1$ is not algebraically $H$-unital since the multiplication map restricted to the algebraic tensor product $\mathcal{L}^1 \odot \mathcal{L}^1$ is not surjective. Since $\mathcal{L}^1$ is topologically $H$-unital, one can use the complex $CC_\bullet(\mathcal{L}^1)$ to compute $HH_\bullet(\mathcal{L}^1)$, without first having to add a unit to $\mathcal{L}^1$. As in [25] we have the trace map $\mathrm{tr}_n\colon (\mathcal{L}^1)^{\hat{\otimes}_\pi n} \to \mathbb{C}^{\hat{\otimes}_\pi n}$ defined by

$$\mathrm{tr}\,(A \otimes B \otimes \cdots \otimes C) = \sum A_{i_0 i_1} \otimes B_{i_1 i_2} \otimes \cdots \otimes C_{i_n i_0} \tag{8}$$

where the sum is extended over all possible sets of indices. In other words $\mathrm{tr}_n(A \otimes B \otimes \cdots \otimes C) = \mathrm{tr}\,(AB \cdots C)$. It is easy to check that $\mathrm{tr}_n$ defines a chain map $\mathrm{tr}_\bullet\colon CC_\bullet(\mathcal{L}^1) \to CC_\bullet(\mathbb{C})$. If $p$ is a rank one projection on $H$ with $\mathrm{tr}\,(p) = 1$ then we have a homomorphism $\mathrm{inc}\colon \mathbb{C} \to \mathcal{L}^1$ which extends to define a chain map $\mathrm{inc}_\bullet\colon CC_\bullet(\mathbb{C}) \to CC_\bullet(\mathcal{L}^1)$. Clearly $\mathrm{tr}_\bullet \circ \mathrm{inc}_\bullet = \mathrm{id}$. However it is also easy to check that the explicit chain homotopy $h$ from $\mathrm{inc}_\bullet \circ \mathrm{tr}_\bullet$ to id given in [25] pages 17–18 goes through in this situation. Therefore we conclude that $\mathrm{tr}_\bullet\colon HH_\bullet(\mathcal{L}^1) \to HH_\bullet(\mathbb{C})$ is an isomorphism and hence we have the following description of the Hochschild, cyclic and periodic cyclic homology groups of $\mathcal{L}^1$.

**Proposition 3.3 ([2,15])** *The Hochschild homology groups $HH_n(\mathcal{L}^1)$ are all zero, except for $n = 0$ when we have $HH_0(\mathcal{L}^1) = \mathbb{C}$. It follows therefore that we have*

$$HC_n(\mathcal{L}^1) = \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

*Finally the periodic cyclic homology groups $HP_n(\mathcal{L}^1)$ equal $\mathbb{C}$ if $n$ is even and are zero if $n$ is odd.*

More generally, one can show the following

**Lemma 3.4** *If $\mathcal{A}$ is a unital Fréchet algebra then $\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1$ is topologically $H$-unital.*

The continuous $\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1$-linear section is obtained from (7) by extending it $\mathcal{A}$-linearly. In particular this Lemma has as a consequence that the $C^\infty(X, \mathcal{L}^1) = C^\infty(X) \otimes_\pi \mathcal{L}^1$ is topologically $H$-unital.

We can define a generalised trace morphism $\mathrm{tr}_\bullet\colon CC_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \to CC_\bullet(\mathcal{A})$ using a generalisation of the formula (8) above — this is because we can identify $(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1)^{\hat{\otimes}_\pi k} = \mathcal{A}^{\hat{\otimes}_\pi k}\hat{\otimes}_\pi(\mathcal{L}^1)^{\hat{\otimes}_\pi k}$ and then use (8) to define a map $\mathcal{A}^{\hat{\otimes}_\pi k}\hat{\otimes}_\pi(\mathcal{L}^1)^{\hat{\otimes}_\pi k}$
$\to \mathcal{A}^{\hat{\otimes}_\pi k}$ which is compatible with $b$ and $B$. The proof of [25] referred to above also extends to this situation to show that $\mathrm{tr}_\bullet \circ \mathrm{inc}_\bullet = \mathrm{id}$ and $\mathrm{inc}_\bullet \circ \mathrm{tr}_\bullet = \mathrm{id}$ where $\mathrm{inc}_\bullet\colon CC_\bullet(\mathcal{A}) \to CC_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1)$ is induced by the homomorphism

$a \mapsto a \otimes p$ where $p$ is a projection of rank one in $H$, normalised to have trace $\mathrm{tr}\,(p) = 1$. Thus we have the following generalisation of Proposition 3.3.

**Proposition 3.5** *The generalised trace morphism* $tr_\bullet \colon CC_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \to CC_\bullet(\mathcal{A})$ *induces isomorphisms* $HH_n(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \cong HH_n(\mathcal{A})$, $HC_n(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \cong HC_n(\mathcal{A})$ *and* $HP_n(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \cong HP_n(\mathcal{A})$ *in Hochschild, cyclic and periodic cyclic homology respectively.*

**Remark 3.6** *This Proposition can be viewed as a version of topological Morita invariance of Hochschild, cyclic and periodic cyclic homology, where the finite dimensional matrices are replaced by their completion in the* $\mathcal{L}^1$ *norm. It also has as a consequence that the trace morphism* $tr_n \colon HH_n(C^\infty(X,\mathcal{L}^1)) \to HH_n(C^\infty(X))$*, when combined with Connes' HKR map* (6) *above, gives an identification of the Hochschild homology of the algebra* $C^\infty(X,\mathcal{L}^1)$ *with the space of differential forms on* $X$*.*

We also mention the following result of Cuntz examining the effect in periodic cyclic homology of replacing $\mathcal{L}^1$ by $\mathcal{L}^p$ for $p \geq 1$.

**Proposition 3.7 ([15])** *If* $1 \leq p$*, then the homomorphism* $\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1 \to \mathcal{A}\hat{\otimes}_\pi\mathcal{L}^p$ *induced by the inclusion* $\mathcal{L}^1 \hookrightarrow \mathcal{L}^p$ *induces isomorphism* $HP_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^1) \stackrel{\cong}{\to} HP_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^q)$ *in periodic cyclic homology.*

When this is combined with Proposition 3.5, we deduce that $HP_\bullet(\mathcal{A}) \stackrel{\cong}{\to} HP_\bullet(\mathcal{A}\hat{\otimes}_\pi\mathcal{L}^p)$ for all $p \geq 1$, and therefore can be viewed as yet another version of topological Morita invariance periodic cyclic homology, where the finite dimensional matrices are replaced by the topological completion given by $\mathcal{L}^p$.

*For the rest of the paper, we will assume that* $X$ *is a compact manifold, even if not explicitly stated.*

The following generalizes Lemma 3.4.

**Lemma 3.8** *Let* $X$ *be a compact manifold. Then the* $*$*-algebra* $C^\infty(\mathcal{L}^1(P))$ *is topologically* $H$*-unital.*

**Proof.** Let $\{U_\alpha\}$ be an open cover by good open subsets of $X$, and $\{\phi_\alpha\}$ be a partition of unity such that $\sum_\alpha \phi_\alpha^2 = 1$. Let $\mathcal{L}^1(P)_\alpha = \mathcal{L}^1(P))\Big|_{U_\alpha}$ be the restriction, and

$$r_\alpha : C^\infty(\mathcal{L}^1(P)) \to C^\infty(\mathcal{L}^1(P)_\alpha)$$

be the induced restriction homomorphism. Let

$$i_\alpha : C^\infty(\mathcal{L}^1(P)_\alpha) \to C^\infty(\mathcal{L}^1(P))$$

be defined as $i_\alpha(g) = \phi_\alpha g$. Note that $i_\alpha(r_\alpha(f)g) = fi_\alpha(g)$ for $f \in C^\infty(\mathcal{L}^1(P))$. By Proposition 3.5 and Remark 3.6, we know that $C^\infty(\mathcal{L}^1(P)_\alpha) = C^\infty(U_\alpha, \mathcal{L}^1)$ is topologically H-unital, therefore there is a continuous left $C^\infty(\mathcal{L}^1(P)_\alpha)$-linear section of the multiplication map,

$$s_\alpha : C^\infty(\mathcal{L}^1(P)_\alpha) \to C^\infty(\mathcal{L}^1(P)_\alpha) \otimes_\pi C^\infty(\mathcal{L}^1(P)_\alpha).$$

Define the map

$$t_\alpha : C^\infty(\mathcal{L}^1(P)) \to C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(P))$$

as the composition $t_\alpha = (i_\alpha \otimes i_\alpha) \circ s_\alpha \circ r_\alpha$. Note that $t_\alpha$ is not quite a section for the multiplication map, instead we have $m \circ t_\alpha = \phi_\alpha^2 t_\alpha$. Now define $t = \sum_\alpha t_\alpha$, so that we get a map

$$t : C^\infty(\mathcal{L}^1(P)) \to C^\infty(\mathcal{L}^1(P)) \otimes_\pi C^\infty(\mathcal{L}^1(P)).$$

It is easy to check that $t$ is a continuous left $C^\infty(\mathcal{L}^1(P))$ section of the multiplication map, establishing the topological H-unitality of $C^\infty(\mathcal{L}^1(P))$.

## 4 Twisted $K$-theory and twisted cohomology

### *4.1 Twisted $K$-theory*

Let $X$ be a compact space which comes equipped with a principal $PU$ bundle $P \to X$ where $PU = PU(H)$ is the projective unitary group of a separable Hilbert space $H$. We will assume that the topology on $PU$ is induced from the norm topology on the full unitary group $U = U(H)$ (this has the advantage that $PU$ acquires the structure of a Banach Lie group, this would not be true for the topology on $PU$ induced from the strong operator topology on $U$ — see § 4.4 for more details). $PU$ acts by conjugation on the space $\mathcal{K} = \mathcal{K}(H)$ of compact operators on $H$. Hence we can form the associated bundle $\mathcal{K}(P)$ and consider the (non-unital) $C^*$-algebra $A = C(\mathcal{K}(P))$ of continuous sections of $P \times_{PU} \mathcal{K}$. The $K$-theory $K_\bullet(A)$ of the algebra $A$ is called the *twisted $K$-theory* of the pair $(X, P)$ [34] and is denoted $K^i(X, P)$. The groups $K^i(X; P)$ are a functor from the category of spaces equipped with a principal $PU$ bundle to the category of abelian groups. In this respect the groups $K^i(X; P)$ have many similarities with the groups $K^i(X)$, for example there are long exact sequences associated to pairs $(X, Y)$. In [34] Rosenberg showed that the twisted $K$-groups $K^i(X; P)$ could also be interpreted as vertical homotopy classes of sections of certain bundles of Fredholm operators associated to $P$. There are alternative descriptions of the groups $K^i(X; P)$ — for example in [6] the twisted $K$-groups are interpreted in terms of bundles on $P$ twisted by a gerbe. In [1],

another equivalent definition of twisted $K$-theory is given, where $PU$ is given the compact open topology instead of the norm topology.

### 4.2 Smooth subalgebras

In this note however we want to study the $K$-theory of the algebra $A$ from the point of view of non-commutative geometry. We must therefore find a smooth replacement for $A$. Assume from now on that $X$ is a smooth compact manifold and that $P \to X$ is a Banach principal $PU$ bundle (so that $P$ is a Banach manifold).

We begin by recalling some generalities on smooth subalgebras of $C^*$-algebras. Let $A$ be a $C^*$-algebra and $\widetilde{A}$ be obtained by adjoining a unit to $A$. Let $\mathcal{A}$ be a $*$-subalgebra of $A$ and $\widetilde{\mathcal{A}}$ be obtained by adjoining a unit to $\mathcal{A}$. Then $\mathcal{A}$ is said to be a *smooth subalgebra* of $A$ if the following two conditions are satisfied:

(1) $\mathcal{A}$ is a dense $*$-subalgebra of $A$;
(2) $\mathcal{A}$ is stable under the holomorphic functional calculus, that is, for any $a \in \widetilde{\mathcal{A}}$ and for any function $f$ that is holomorphic in a neighbourhood of the spectrum of $a$ (thought of as an element in $\widetilde{A}$) one has $f(a) \in \widetilde{\mathcal{A}}$.

Assume that $\mathcal{A}$ is a dense $*$-subalgebra of $A$ such that $\mathcal{A}$ is a Fréchet algebra with a topology that is finer than that of $A$. A necessary and sufficient condition for $\mathcal{A}$ to be a smooth subalgebra is given by the *spectral invariance* condition cf. [35, Lemma 1.2]:

- $\widetilde{\mathcal{A}} \cap GL(\widetilde{A}) = GL(\widetilde{\mathcal{A}})$, where $GL(\widetilde{\mathcal{A}})$ and $GL(\widetilde{A})$ denote the group of invertibles in $\widetilde{\mathcal{A}}$ and $\widetilde{A}$ respectively.

Now $\mathcal{L}^p$ is a dense $*$-subalgebra of $\mathcal{K}$. Suppose that $1 + a \in \tilde{\mathcal{L}}^p$ is such that $1 + a \in GL(\tilde{\mathcal{K}})$, i.e. there is $1 + b \in GL(\tilde{\mathcal{K}})$ such that $(1 + a)(1 + b) = 1$. Then $a + b + ab = 0$, so that $b = -a - ab \in \mathcal{L}^p$, i.e $1 + b \in GL(\tilde{\mathcal{L}}^p)$. This shows that $\mathcal{L}^p$ is a spectral invariant subalgebra of $\mathcal{K}$. It follows that $\mathcal{L}^p$ is a smooth dense subalgebra of $\mathcal{K}$.

Let $\mathcal{L}^1(P) = P \times_{PU} \mathcal{L}^1$ be the associated algebra bundle on $X$ (here $PU$ acts on $\mathcal{L}^1$ by conjugation). Let $\mathcal{A} = C^\infty(\mathcal{L}^1(P))$ be the $*$-algebra of smooth sections of $\mathcal{L}^1(P)$, which is made into a Fréchet algebra with the topology defined in §2.

Let $\mathfrak{A} = C(\mathcal{L}^1(P))$ be the $B^*$-algebra of continuous sections. $\mathfrak{A}$ is then a smooth, dense subalgebra of $A = C(\mathcal{K}(P))$. This can be seen as follows. The argument given above shows that if $1 + a \in \tilde{\mathfrak{A}}$ is such that $1 + a \in GL(\tilde{A})$, then $1 + a \in GL(\tilde{\mathfrak{A}})$. Next, a direct generalization of the approximation argument

showing that $C^\infty(X)$ is a smooth dense subalgebra of $C(X)$ shows that $\mathcal{A}$ is a smooth dense subalgebra of $\mathfrak{A}$. Combining these observations, we see that $\mathcal{A}$ is a smooth dense subalgebra of $A$ as desired and it also follows that $\mathcal{A}$ is stable under the holomorphic functional calculus. Therefore by a result in the appendix in [11], (a more detailed proof is given in [5], see also chapter 3, [23]) the inclusion map $\mathcal{A} \subset A$ induces an isomorphism $K_i(\mathcal{A}) \cong K_i(A)$.

### *4.3 Connes-Chern character*

Recall how the Connes-Chern character

$$\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$$

is constructed. Let $\mathcal{C}$ be a unital $*$-algebra with unit $1_\mathcal{C}$, and let $\mathrm{Proj}(\mathcal{C})$ be its set of self-adjoint projections. Two projections $P, Q \in \mathrm{Proj}(\mathcal{C})$ are said to be *Murray-von Neumann equivalent* if there is an element $V \in \mathcal{C}$ such that $P = V^*V$ and $Q = VV^*$. Denote $M_n(\mathbb{C}) \otimes \mathcal{C} = M_n(\mathcal{C})$, where $M_n(\mathbb{C})$ denotes the square matrices of size $n$ over $\mathbb{C}$. Then $M_n(\mathcal{C})$ is also a $*$-algebra. Let $M_\infty(\mathcal{C}) = \lim_{n\to\infty} M_n(\mathcal{C})$ be the direct limit of the embeddings of $M_n(\mathcal{C})$ in $M_{n+1}(\mathcal{C})$ given by $T \to \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$. Let $V(\mathcal{C}) = \mathrm{Proj}(M_\infty(\mathcal{C}))/\sim$ denote the Murray-von Neumann equivalence classes of projections in $M_\infty(\mathcal{C})$. Then $V(\mathcal{C})$ is an Abelian semi-group under with the operation induced by the direct sum, and the associated Abelian group is called the Grothendieck group $K_0(\mathcal{C})$. The homomorphism $\pi : \mathbb{C} \to \mathcal{C}$ given by $\lambda \mapsto \lambda \cdot 1_\mathcal{C}$ induces a homomorphism $\pi_* : K_0(\mathbb{C}) \cong \mathbb{Z} \to K_0(\mathcal{C})$. Then the reduced $K$-group $\tilde{K}_0(\mathcal{C})$ is defined as $\tilde{K}_0(\mathcal{C}) = \operatorname{coker} \pi_* \cong K_0(\mathcal{C})/\mathbb{Z}$.

Now $\mathcal{A}$ is a non-unital $*$-algebra; let $\widetilde{\mathcal{A}}$ be the unitization of $\mathcal{A}$. By definition, the $K$-group $K_i(\mathcal{A})$ is the reduced $K$-group $\tilde{K}_i(\widetilde{\mathcal{A}})$ of $\tilde{\mathcal{A}}$. Explicitly, let $P, Q$ be projections in $M_n(\widetilde{\mathcal{A}})$ such that $P - Q \in M_n(\mathcal{A})$. Then $[P - Q] \in K_0(\mathcal{A})$. Recall that the Connes-Chern character is defined as

$$\mathrm{ch}([P-Q]) = \mathrm{tr}\,(P-Q) + \sum_{n\in\mathbb{N}} (-u)^n \frac{(2n)!}{n!}\ \mathrm{tr}\left(\left(P - Q - \frac{1}{2}\right)(d(P-Q))^{2n}\right), \tag{9}$$

which is an element of the complex $CC_k(\mathcal{A}) \otimes \mathbb{C}((u))$ with boundary operator $b + uB$. Now let $U$ be an invertible element in $M_n(\widetilde{\mathcal{A}})$ such that $U - 1 \in M_n(\mathcal{A})$. Then $[U] \in K_1(\mathcal{A})$, and the odd Connes-Chern character is defined as

$$\mathrm{ch}([U]) = \sum_{n\geq 0} u^n n!\ \mathrm{tr}\left(\left(U^{-1} - 1\right) d(U-1)(d(U^{-1}-1)d(U-1))^n\right), \tag{10}$$

which is an element of the complex $CC_\bullet(\mathcal{A}) \otimes \mathbb{C}((u))$ with boundary operator

$b + uB$.

### 4.4 The Dixmier-Douady Class

We consider principal bundles $P$ on $X$ with structure group $PU$. Isomorphism classes of such bundles on $X$ are in bijective correspondence with $H^3(X;\mathbb{Z})$. To see this recall that by Kuiper's theorem $PU$ has the homotopy type of a $K(\mathbb{Z},2)$ and hence $BPU$ has the homotopy type of a $K(\mathbb{Z},3)$. Recall that we are considering $PU$ with the topology induced from the norm topology on $U$. Recall that $U = U(H)$ is a Banach Lie group (see for example [33]). $U(1)$ is a closed subgroup of $U$ and hence $PU$ has a natural structure as a Banach Lie group. This is not the case for the strong operator topology on $PU$; $PU$ equipped with the strong operator topology is an example of a *Polish* group. Note that it is possible to realise $BPU$ as a Banach manifold — see [10] for more details. We associate a Čech class $\delta(P) \in H^3(X;\mathbb{Z})$ to a principal $PU$ bundle $P$ on $X$ by first choosing a good open cover $\{U_i\}_{i\in I}$ of $X$ relative to which $P$ has $PU$-valued transition functions $g_{ij}$ (recall that a good cover is one for which every non-empty finite intersection $U_{i_1} \cap \cdots \cap U_{i_p}$ is contractible). Choose lifts $\hat{g}_{ij}\colon U_{ij} \to U$ of $g_{ij}$ and define $\epsilon_{ijk}\colon U_{ijk} \to U(1)$ by $\hat{g}_{ij}\hat{g}_{jk} = \hat{g}_{ik}\epsilon_{ijk}$. Since $U(1)$ is central in $U$ one can show that $\epsilon_{ijk}$ satisfies the Čech 2-cocycle condition. Finally we choose maps $w_{ijk}\colon U_{ijk} \to \mathbb{R}$ such that $\exp(2\pi i w_{ijk}) = \epsilon_{ijk}$ and define $n_{ijkl} = w_{jkl} - w_{ikl} + w_{ijl} - w_{ijk}$. Then $n_{ijkl}$ is a Čech representative for a class $\delta(P) \in H^3(X;\mathbb{Z})$.

Freed in [18] gives a very nice construction of a differential 3-form representing the image in de Rham cohomology of the Dixmier-Douady class $\delta(P)$ (note that a similar construction appeared in [20] and in the context of loop groups in [29]). We recall his result here, phrased in the language of Michael Murray's *bundle gerbes* [28]. Associated to $P$ is the *lifting bundle gerbe* $L \to P^{[2]}$, where $P^{[2]} = P \times_\pi P$ is the fibre product. Recall that $L$ is the complex line associated to the principal $U(1)$ bundle on $P^{[2]}$ obtained by pulling back the universal $U(1)$ bundle on $PU$ via the canonical map $g\colon P^{[2]} \to PU$. Here $g(p_1,p_2)$ for $(p_1,p_2) \in P^{[2]}$ is the unique element of $PU$ such that $p_2 = p_1 g(p_1,p_2)$. The product on the group $U$ induces a *bundle gerbe product* on $L$, i.e. a line bundle isomorphism which on fibres takes the form $L_{(p_1,p_2)} \otimes L_{(p_2,p_3)} \to L_{(p_1,p_3)}$ for points $p_1$, $p_2$ and $p_3$ of $P$ all lying in the same fibre of $P$ over $X$. A *bundle gerbe connection* on $L$ is a connection $\nabla_L$ on the line bundle $L$ which is preserved by the bundle gerbe product. Thus if $s$ and $t$ are sections of $L$ then we have $\nabla_L(st)(p_1,p_3) = \nabla_L(s)(p_1,p_2)t(p_2,p_3) + s(p_1,p_2)\nabla_L(t)(p_2,p_3)$. It can be shown that the curvature $F_{\nabla_L}$ of a bundle gerbe connection $\nabla_L$ can always be written in the form $F_{\nabla_L} = \pi_1^* f - \pi_2^* f$ for some 2-form $f$ on $P$, where $\pi_1$ and $\pi_2$ denote the projections onto the first and second factors in $P^{[2]}$ respectively. It can also be shown that $\mathrm{d}f$ is a basic form, i.e. $\mathrm{d}f = \pi^*\omega$ for some necessarily

closed 3-form $\omega$ on $X$. For more details we refer to [28]. $\omega$ is a representative for the image in de Rham cohomology of the Dixmier-Douady class $\delta(P)$ of $P$.

Write $\mathfrak{u}$ and $\mathfrak{pu}$ for the Lie algebras of $U$ and $PU$ respectively. Denote by $\mathrm{Split}(\mathfrak{u}, \mathfrak{pu})$ the set of splittings $\sigma\colon \mathfrak{pu} \to \mathfrak{u}$ of the central extension of Lie algebras $i\mathbb{R} \to \mathfrak{u} \xrightarrow{p} \mathfrak{pu}$. Note that $\mathrm{Split}(\mathfrak{u}, \mathfrak{pu})$ is an affine space. $PU$ acts on $\mathrm{Split}(\mathfrak{u}, \mathfrak{pu})$ by $g \cdot \sigma = Ad(\hat{g}^{-1})\sigma Ad(g)$ where $\hat{g} \in U$ is such that $p(\hat{g}) = g$. Therefore we can form the associated bundle $\mathrm{Split}(P)$ on $X$. Since the fibre of $\mathrm{Split}(P)$ is affine we can choose a section which we denote by $\sigma$. Observe that a splitting $s$ of the projection $\mathfrak{u} \xrightarrow{p} \mathfrak{pu}$ determines a left invariant connection $\hat{\theta}_L - s(\theta_L)$ on the principal $U(1)$ bundle $U \to PU$ with curvature $\frac{1}{2}[s(\theta_L), s(\theta_L)] - \frac{1}{2}s[\theta_L, \theta_L]$ (here $\hat{\theta}_L$ and $\theta_L$ denote the left Maurer-Cartan forms on $U$ and $PU$ respectively). Clearly then $\sigma$ induces a connection on the pullback principal $U(1)$ bundle on $P^{[2]}$ and hence a connection $\nabla_L$ on the associated line bundle $L$. $\nabla_L$ is given locally by $\nabla_L = \mathrm{d} + A_\alpha$ where $A_\alpha$ is the local 1-form $\hat{g}_\alpha^*\hat{\theta}_L - \pi_2^*\sigma(g^*\theta_L)$ ($\hat{g}_\alpha$ is a local lift of $g\colon P^{[2]} \to PU$ to $U$). The equivariance property of $\sigma$ under the action of $PU$ shows that $\nabla_L$ is a bundle gerbe connection. The curvature $F_{\nabla_L}$ of $\nabla_L$ is easily computed to be $F_{\nabla_L} = \mathrm{d}\hat{g}^*\theta - \pi_2^*\mathrm{d}\sigma(g^*\theta)$. Now suppose we have chosen a connection $\Theta$ on the principal $PU$ bundle $P \to X$ with curvature $\Omega$. A calculation, using the identity $g^*\Theta = \mathrm{Ad}(g^{-1})\Theta + g^*\theta_L$, shows that the 2-form $f$ on $P$ defined by

$$f = \mathrm{d}\sigma(\Theta) + \frac{1}{2}[\sigma(\Theta), \sigma(\Theta)] - \sigma(\mathrm{d}\Theta + \frac{1}{2}[\Theta, \Theta]) \tag{11}$$

satisfies $F_{\nabla_L} = \pi_1^*f - \pi_2^*f$. The results of Murray [28] show that $\mathrm{d}f = \pi^*c(P)$ for some necessarily closed 3-form $c(P)$ on $X$. In fact $c(P)$ is the push-forward of the basic 3-form on $P$ given by

$$[\sigma(\Omega), \sigma(\Theta)] - \sigma([\Omega, \Theta]) - \sigma'(\Omega) \tag{12}$$

where we think of $\sigma$ as a $PU$ equivariant map $P \to \mathrm{Split}(\mathfrak{u}, \mathfrak{pu})$ and denote by $\sigma'$ its derivative. $c = c(P)$ is a de Rham cocycle representative for the image of $\delta(P)$ in real cohomology. For later use we need to know how the 3-form $c$ depends on the choice of connection $\Theta$ on $P$ (of course there is also a dependence on the choice of section $\sigma$ but this will not be so important in the sequel). When we want to avoid confusion about the choice of connection $\Theta$ on $P$ used to construct $c$ we will write $c = c(\Theta)$. Any two connections $\Theta$ and $\Theta'$ on $P$ differ by a 1-form $A$ on $X$ with values in the bundle $\mathrm{ad}\, P$. Let $f'$ be the 2-form (11) constructed from the connection $\Theta'$ (same choice of $\sigma$). We have $f' = f + \pi^*\beta$ where $\beta$ is the 2-form on $X$ given by

$$\beta = [\sigma(\Theta), \sigma(A)] - \sigma([\Theta, A]) + \frac{1}{2}[\sigma(A), \sigma(A)] - \frac{1}{2}\sigma([A, A]) + \sigma'(A) \tag{13}$$

Note that $\sigma'(A)$ is a scalar valued 2-form and therefore (13) does define a

2-form on $X$. It follows therefore that we have $c(\Theta') = c(\Theta) + \mathrm{d}\beta$ on $X$.

### *4.5 Twisted Cohomology*

In [6] (see also [19]) it was argued that the Chern character in twisted $K$-theory took values in a certain 'twisted' cohomology group. If $X$ is a smooth manifold then this group can be defined as follows. Let $c = c(P)$ be a de Rham cocycle representative for $\delta(P)$. Thus $c$ is a closed 3-form on $X$ with integer periods. Let $\Omega^\bullet(X) = \bigoplus_{n=0} \Omega^n(X)$ denote the graded algebra of differential forms on $X$. As above we introduce a formal variable $u$ of degree $-2$ and consider the graded algebras $\Omega^\bullet(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$ and $\Omega^\bullet(X) \otimes \mathbb{C}((u))$ consisting of differential forms on $X$ with values in formal power series in the variable $u^{-1}$ and formal Laurent series in $u$ respectively. We can equip both of these graded algebras with a differential of degree one given by $\mathrm{d} - uc$. Thus this differential is a 'twisting' of the ordinary de Rham differential d by exterior multiplication with the closed 3-form $c$, scaled by the variable $u$ so as to have degree one.

**Definition 1** *We call the homology of the complex* $(\Omega^\bullet(X) \otimes \mathbb{C}((u)), d - uc)$ *the* twisted de Rham cohomology *of $X$ (with respect to $c$) and denote it by $H^\bullet(X; c)$.*

If $c$ changes by a coboundary, $c' = c + \mathrm{d}\beta$, then the complexes $(\Omega^\bullet(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]], \mathrm{d} - uc)$ and $(\Omega^\bullet(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]], \mathrm{d} - uc')$ are chain isomorphic through the chain map given by multiplication with $e^{u\beta}$. Similarly, the complexes $(\Omega^\bullet(X) \otimes \mathbb{C}((u)), \mathrm{d} - uc)$ and $(\Omega^\bullet(X) \otimes \mathbb{C}((u)), \mathrm{d} - uc')$ are also chain isomorphic through multiplication by $e^{u\beta}$.

If $\{U_i\}_{i\in I}$ is a good cover of the manifold $X$ then $H^\bullet(X;c)$ can be computed from the following 'twisted Čech-de Rham' double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \\
\bigoplus_i \Omega^1(U_i)((u)) & \xrightarrow{\delta} & \bigoplus_{i<j} \Omega^1(U_{ij})((u)) & \xrightarrow{\delta} & \bigoplus_{i<j<k} \Omega^1(U_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \\
\bigoplus_i \Omega^0(U_i)((u)) & \xrightarrow{\delta} & \bigoplus_{i<j} \Omega^0(U_{ij})((u)) & \xrightarrow{\delta} & \bigoplus_{i<j<k} \Omega^0(U_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \\
\bigoplus_i \Omega^{-1}(U_i)((u)) & \xrightarrow{\delta} & \bigoplus_{i<j} \Omega^{-1}(U_{ij})((u)) & \xrightarrow{\delta} & \bigoplus_{i<j<k} \Omega^{-1}(U_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \uparrow{\scriptstyle \mathrm{d}-uc} & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{14}$$

Here by $\Omega^n(X)((u))$ we mean the elements of $\Omega^\bullet(X)\otimes\mathbb{C}((u))$ of total degree $n$, so that $\Omega^n(X)((u)) = \oplus_{p+q=n}\Omega^p(X)u^q$. Associated to this double complex are two spectral sequences (corresponding to the two filtrations of the total complex by rows and by columns) which compute the cohomology of the total complex. If we filter by rows then the $E_1$ term of the associated spectral sequence has $E_1^{0,q} = \Omega^q(X)((u))$ and $E_1^{p,q} = 0$ for $p > 0$. Therefore the spectral sequence collapses at the $E_2$ term and computes the twisted de Rham cohomology of $X$. However if we filter by columns then the resulting spectral sequence has $E_2^{p,q} = H^p(X)u^q$ for $q$ even and $E_2^{p,q} = 0$ for $q$ odd. Further one can show that the differential $d_3\colon E_3^{p,q} \to E_3^{p+3,q-2}$ is cup product with $[c]u$. We summarise this discussion in the following proposition.

**Proposition 4.1** *There exists a spectral sequence converging to $H^\bullet(X;c)$ with $E_2^{p,q} = H^p(X;\mathbb{C})u^q$ if $q$ is even and $E_2^{p,q} = 0$ if $q$ is odd. The even differentials $d_{2r}$ are all zero and the differential $d_3\colon E_3^{p,q} \to E_3^{p+3,q-2}$ is cup product with $[c]u$.*

Note that a similar spectral sequence was obtained in [19] by different means.

**Remark 4.2** *In this next section, we are going to relate periodic cyclic homology to twisted de Rham cohomology. For this reason we like to think of twisted de Rham cohomology as the* homology *groups of a complex and we will therefore equip the complex $\Omega^\bullet(X)\otimes\mathbb{C}((u))$ with a differential of degree $-1$ given by $ud - u^2c$. The only effect of this change is that the homology groups*

*$H_p(\Omega^\bullet(X)\otimes\mathbb{C}((u)))$ of the complex $\Omega^\bullet(X)\otimes\mathbb{C}((u))$ equipped with the differential $ud-u^2c$) are equal to $u$ times the twisted cohomology groups $H^p(X,c)$. For this reason we will still refer to $H_p(\Omega^\bullet(X)\otimes\mathbb{C}((u)))$ as the twisted de Rham cohomology of $X$.*

## 5 Periodic Cyclic Homology and Twisted Cohomology

### *5.1 A Chain Map*

In [21,22] Gorokhovsky introduces the notion of a generalised cycle and defines the character of such an object, which turns out to be a cocycle in the $(b,B)$-bicomplex computing cyclic cohomology. We will use Gorokhovsky's formalism to construct a map from the $(b,B)$-bicomplex computing cyclic homology to the twisted de Rham complex.

Define bundles $\mathcal{L}^1(P)$ and $\mathcal{B}(P)$ associated to $P$ via the adjoint action of $PU$ on $\mathcal{L}^1$ and $\mathcal{B}(H)$ respectively. From now on let us denote by $\mathcal{A}$ the Fréchet algebra $C^\infty(\mathcal{L}^1(P))$. Choose a connection $\Theta$ on the principal bundle $P$ and let $c=c(\Theta)$ be the associated 3-form (12). Define a connection $\nabla$ on $\mathcal{L}^1(P)$ by $\nabla(s)=\mathrm{d}s+[\sigma(\Theta),s]$ for a section $s\in C^\infty(\mathcal{L}^1(P))$. $\nabla$ acts as a derivation of degree 1 on the graded algebra $\Omega^\bullet(\mathcal{L}^1(P))=\bigoplus_{n=0}\Omega^n(\mathcal{L}^1(P))$ of forms on $X$ with values in $\mathcal{L}^1(P)$. Let $\Omega$ be the curvature of the connection $\Theta$ on the principal $PU$ bundle $P\to X$. $\sigma(\Omega)$ can be identified as a section of the bundle $\Omega^2(\mathcal{B}(P)))$. Since $\mathcal{L}^1$ is an ideal in $\mathcal{B}(H)$, there is an action of $\Omega^\bullet(\mathcal{B}(P))$ on $\Omega^\bullet(\mathcal{L}^1(P))$. In particular, $\sigma(\Omega)$ acts as a multiplier on the graded algebra $\Omega^\bullet(\mathcal{L}^1(P))$. Note that $\nabla(\sigma(\Omega)s)=\sigma(\Omega)\nabla(s)-cs$ where $c$ acts on $s$ by the obvious action of $\Omega^\bullet(X)$ on $\Omega^\bullet(\mathcal{L}^1(P))$. Here $c$ is the 3-form representing the Dixmier-Douady class of the bundle $P$ obtained in (12). Finally observe that the ordinary operator trace tr defines a trace map $\mathrm{tr}:\Omega^\bullet(\mathcal{L}^1(P))\to\Omega^\bullet(X)$ satisfying the properties that $\mathrm{tr}\,(s_1s_2)=(-1)^{|s_1||s_2|}\mathrm{tr}\,(s_2s_1)$ for $s_1,s_2\in\Omega^\bullet(\mathcal{L}^1(P))$ and $\mathrm{tr}\,(\nabla(s))=\mathrm{d}\mathrm{tr}\,(s)$ for all $s\in\Omega^\bullet(\mathcal{L}^1(P))$.

**Definition 2** *Following Gorokhovsky [21] define a map $\mathrm{Ch}_k\colon CC_k(\mathcal{A})\to\Omega^\bullet(X)\otimes\mathbb{C}((u))$ by the JLO-type formula*

$$\mathrm{Ch}(\tilde a_0,a_1,\dots,a_k)=\int_{\Delta^k} tr\,(\tilde a_0e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k. \tag{15}$$

*Note that $\mathrm{Ch}_k$ is of degree zero. In addition we also put $\mathrm{Ch}_0(a_0)=tr\,(a_0e^{-\sigma(\Omega)u})$. $\mathrm{Ch}_k$ extends to a $\mathbb{C}((u))$-module map $\mathrm{Ch}\colon CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u))\to\Omega^\bullet(X)\otimes\mathbb{C}((u))$ and descends to the quotient to give a map (also written $\mathrm{Ch}$) $\mathrm{Ch}\colon CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u))/u\mathbb{C}[[u]]\to\Omega^\bullet(X)\otimes\mathbb{C}((u))/u\mathbb{C}[[u]]$. Finally, setting $u=0$ defines a*

*map* $\mathrm{Ch}\colon CC_\bullet(\mathcal{A}) \to \Omega^\bullet(X)$ *given by*

$$\mathrm{Ch}(\tilde{a}_0, a_1, \ldots, a_k) = \frac{1}{k!} tr\,(\tilde{a}_0 \nabla(a_1)\cdots\nabla(a_k)). \tag{16}$$

**Remark 5.1** *If the bundle $P$ is trivial, and we take for the connection $\nabla$ on $\mathcal{L}^1(P)$ the trivial connection $d$ then the map* Ch *reduces to the expression* (6)*.*

**Remark 5.2** *Note that the map* $\mathrm{Ch}\colon CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u)) \to \Omega^\bullet(X)\otimes\mathbb{C}((u))$ *depends on a choice of connection $\nabla$ on the bundle $\mathcal{L}^1(P)$. When we want to make this precise we will write* $\mathrm{Ch} = \mathrm{Ch}(\nabla)$*.*

**Remark 5.3** *In the formula* (15) *above we understand that the expression appearing as the integrand,* $\tilde{a}_0 e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u}$*, is to mean*

$$a_0 e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u} + \lambda e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u}$$

*if* $\tilde{a}_0 = (a_0, \lambda)$*.*

Each of the maps Ch defined above is a morphism of complexes. To see this we need the following Proposition.

**Proposition 5.4** *For each of the maps* Ch *defined above, we have*

$$\mathrm{Ch}\circ(b + uB) = (u\mathrm{d} - u^2 c)\circ\mathrm{Ch}. \tag{17}$$

**Proof.** We first compute $\mathrm{dCh}(\tilde{a}_0, a_1, \ldots, a_k)$. This equals

$$\begin{aligned}
&\int_{\Delta^k} \mathrm{tr}\,(\nabla(a_0)e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k \\
&\qquad + cu\int_{\Delta^k} \mathrm{tr}\,(a_0 e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k \\
&\qquad + \sum_{i=1}^k (-1)^{i-1}\int_{\Delta^k} \mathrm{tr}\,(\tilde{a}_0 e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots e^{-s_{i-1}\sigma(\Omega)u}[\sigma(\Omega), a_i] \\
&\qquad\qquad\qquad e^{-s_i\sigma(\Omega)u}\nabla(a_{i+1})\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k,
\end{aligned}$$

where we use the fact that $\nabla(\sigma(\Omega)) = -c$ and $\nabla^2(a) = [\sigma(\Omega), a]$. Therefore we have that $(u\mathrm{d} - u^2c)\circ\mathrm{Ch}(\tilde{a}_0, a_1, \ldots, a_k)$ is equal to the sum of the two terms,

$$u\int_{\Delta^k} \mathrm{tr}\,(\nabla(a_0)e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k, \tag{18}$$

and

$$\begin{aligned}
&\sum_{i=1}^k (-1)^{i-1} u\int_{\Delta^k} \mathrm{tr}\,(\tilde{a}_0 e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots e^{-s_{i-1}\sigma(\Omega)u}[\sigma(\Omega), a_i]e^{-s_i\sigma(\Omega)u} \\
&\qquad\qquad\qquad \nabla(a_{i+1})\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k.
\end{aligned} \tag{19}$$

The first expression (18) is equal to $(\mathrm{Ch} \circ uB)(\tilde{a}_0, a_1, \ldots, a_k)$. To see this, we compute $(\mathrm{Ch} \circ uB)(\tilde{a}_0, a_1, \ldots, a_k)$. We get that this is equal to

$$\begin{aligned}
&\int_{\Delta^{k+1}} \mathrm{tr}\,(e^{-s_0\sigma(\Omega)u}\nabla(a_0)e^{-s_1\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_{k+1}\sigma(\Omega)u})ds_1\cdots ds_{k+1} \\
&+\sum_{i=1}^{k}(-1)^{ik}\int_{\Delta^{k+1}} \mathrm{tr}\,(e^{-s_0\sigma(\Omega)u}\nabla(a_i)e^{-s_1\sigma(\Omega)u}\nabla(a_{i+1})\cdots\nabla(a_k)e^{-s_{k-i+1}\sigma(\Omega)u} \\
&\qquad\qquad \nabla(a_0)e^{-s_{k-i+2}\sigma(\Omega)u}\cdots\nabla(a_{i-1})e^{-s_{k+1}\sigma(\Omega)u})ds_1\cdots ds_{k+1}.
\end{aligned}$$

Commuting differential forms inside the trace gives us that this is equal to

$$\begin{aligned}
&\int_{\Delta^{k+1}} \mathrm{tr}\,(\nabla(a_0)e^{-s_1\sigma(\Omega)u}\nabla(a_1)e^{-s_2\sigma(\Omega)u}\cdots\nabla(a_k)e^{-(s_0+s_{k+1})\sigma(\Omega)u})ds_1\cdots ds_{k+1} \\
&+\sum_{i=1}^{k}\int_{\Delta^{k+1}} \mathrm{tr}\,(\nabla(a_0)e^{-s_{k-i+2}\sigma(\Omega)u}\cdots\nabla(a_{i-1})e^{-(s_0+s_{k+1})\sigma(\Omega)u}\nabla(a_i)e^{-s_1\sigma(\Omega)u} \\
&\qquad\qquad \nabla(a_{i+1})e^{-s_2\sigma(\Omega)u}\cdots\nabla(a_k)e^{-s_{k-i+1}\sigma(\Omega)u})ds_1\cdots ds_{k+1}.
\end{aligned}$$

This is then equal to

$$\begin{aligned}
&\sum_{i=1}^{k}\int_{\Delta^{k}} s_{i-1}\mathrm{tr}\,(\nabla(a_0)e^{-s_0\sigma(\Omega)u}\nabla(a_1)e^{-s_1\sigma(\Omega)u}\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k \\
&\qquad +\int_{\Delta^{k}} s_k\mathrm{tr}\,(\nabla(a_0)e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots\nabla(a_k)e^{-s_k\sigma(\Omega)u})ds_1\cdots ds_k,
\end{aligned}$$

which is equal to (18). We now show that the second expression (19) can be identified with $(\mathrm{Ch}\circ b)(\tilde{a}_0, a_1, \ldots, a_k)$. To do this, we compute $\mathrm{Ch}b(\tilde{a}_0, a_1, \ldots, a_k)$. This is equal to the sum

$$\begin{aligned}
&\sum_{i=1}^{k}(-1)^{i-1}\int_{\Delta^{k-1}} \mathrm{tr}\,(\tilde{a}_0e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots e^{-s_{i-2}\sigma(\Omega)u}\nabla(a_{i-1}) \\
&\qquad\qquad [a_i, e^{-s_{i-1}\sigma(\Omega)u}]\nabla(a_{i+1})\cdots\nabla(a_k)e^{-s_{k-1}\sigma(\Omega)u})ds_1\cdots ds_{k-1}
\end{aligned}$$

As in [32] Equation 7.2 we have the general differentiation formula

$$D(e^K) = \int_0^1 e^{(1-s)K}D(K)e^{sK}ds$$

where $D$ is a derivation. Using this formula for the derivations $[a_i, \cdot\,]$ this equals

$$\begin{aligned}
&u\sum_{i=1}^{k}(-1)^{i+1}\int_{\Delta^{k-1}}\int_0^1 s_{i-1}\mathrm{tr}\,(\tilde{a}_0e^{-s_0\sigma(\Omega)u}\nabla(a_1)\cdots e^{-s_{i-2}\sigma(\Omega)u}\nabla(a_{i-1}) \\
&\qquad e^{-s_{i-1}(1-t)\sigma(\Omega)u}[\sigma(\Omega), a_i]e^{-s_{i-1}t\sigma(\Omega)u}\cdots\nabla(a_k)e^{-s_{k-1}\sigma(\Omega)u})dtds_1\cdots ds_{k-1}
\end{aligned}$$

Changing variables in the integral shows that this expression is equal to (19).

**Remark 5.5** *By looking at the part of equation* (17) *which is of degree zero in* $u$*, we see that the map* (16) *is a chain map from the Hochschild complex* $CC_\bullet(\mathcal{A})$ *to the following complex*

$$0 \to \Omega^0(X) \xrightarrow{0} \Omega^1(X) \xrightarrow{0} \Omega^2(X) \xrightarrow{0} \cdots \tag{20}$$

### *5.2 A Homotopy Formula*

We now want to prove a homotopy formula showing what happens when we perturb the connection $\nabla$. Recall that $\nabla$ depended on a choice of connection $\Theta$ on the principal bundle $P$ and a choice of section $\sigma$ of the associated bundle $\mathrm{Split}(P)$ of splittings. Changing $\Theta$ to a new connection $\Theta'$ will change the twisting cocycle $c(\Theta)$ by a coboundary; i.e. $c' = c + \mathrm{d}\beta$ where $c' = c(\Theta')$ and $c = c(\Theta)$ and hence changes the target space of the map Ch. Recall though that the complexes $(\Omega^\bullet(X)\otimes\mathbb{C}((u)), u\mathrm{d}-u^2c)$ and $(\Omega^\bullet(X)\otimes\mathbb{C}((u)), u\mathrm{d}-u^2c')$ are isomorphic through the chain map $e^{u\beta}$.

Let $\Theta$ and $\Theta'$ be two connections on the prinicipal $PU$ bundle $P \to X$. Then $\Theta$ and $\Theta'$ are related by a 1-form $A$ with values in the bundle $\mathrm{ad}P$: $\Theta' = \Theta + A$. Form the family of connections $\Theta_t = \Theta + tA$ for $0 \le t \le 1$. Let $\Omega_t$ denote the curvature of the connection $\Theta_t$. Denote also by $c_t$ the 3-form (12) associated to the connection $\Theta_t$. The family $\Theta_t$ induces a family of connections $\nabla_t$ on the bundle $\mathcal{L}^1(P)$ by the same formula as above, $\nabla_t(s) = \mathrm{d}s + [\sigma(\Theta_t), s]$ for $s$ a section of $\mathcal{L}^1(P)$. We have $\nabla_t(\Omega_t) = c_t$. Define a 2-form $\beta_t$ on $X$ by the formula

$$\beta_t = t([\sigma(\Theta),\sigma(A)] - \sigma([\Theta,A])) + \frac{t^2}{2}([\sigma(A),\sigma(A)] - \sigma([A,A])) + t\sigma'(A). \tag{21}$$

Note that we have $f_t = f + \pi^*\beta_t$ and hence $c_t = c + \mathrm{d}\beta_t$. Define a connection 1-form $\tilde{\Theta}$ on the principal $PU$ bundle $P \times [0,1] \to X \times [0,1]$ by $\tilde{\Theta} = \Theta_t$. The curvature of the connection $\tilde{\Theta}$ is $\tilde{\Omega} = \Omega_t + \mathrm{d}tA$. If we apply the section $\sigma$ to $\tilde{\Omega}$ we get $\sigma(\tilde{\Omega}) = \sigma(\Omega_t) + \mathrm{d}t\sigma(A)$. Denote by $\tilde{\nabla}$ the connection $\nabla_t + \mathrm{d}t\partial_t$ on the associated bundle $\mathcal{L}^1(P \times [0,1])$ (we will denote differentiation with respect to $t$ by $\partial_t$ and sometimes by a dot). An easy calculation shows that $\tilde{\nabla}\sigma(\tilde{\Omega}) = c + \mathrm{d}\beta_t + \mathrm{d}t\dot{\beta}_t$.

Now we are ready for the homotopy formula. Let us write $\tilde{\mathcal{A}}$ for the algebra $C^\infty(\mathcal{L}^1(P\times[0,1]))$ and $\tilde{\mathrm{Ch}} = \mathrm{Ch}(\tilde{\nabla})$. We have a natural map $\rho\colon \mathcal{A} \to \tilde{\mathcal{A}}$ which sends a section of $\mathcal{L}^1(P)$ to a section of $\mathcal{L}^1(P\times[0,1])$ constant in the $t$ direction. $\rho$ induces a map (also denoted $\rho$) $CC_k(\mathcal{A}) \to CC_k(\tilde{A})$ and hence also maps $\rho\colon CC_k(\mathcal{A})\otimes\mathbb{C}((u)) \to CC_k(\tilde{\mathcal{A}})\otimes\mathbb{C}((u))$ and $\rho\colon CC_k(\mathcal{A})\otimes\mathbb{C}((u))/u\mathbb{C}[[u]] \to CC_k(\tilde{\mathcal{A}}) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$. Notice that we have $\tilde{\nabla}\rho(a) = \nabla_t\rho(a)$ for $a \in \mathcal{A}$ and hence $\tilde{\mathrm{Ch}}\rho|_{t=1} = \mathrm{Ch}(\nabla')$, $\tilde{\mathrm{Ch}}\rho|_{t=0} = \mathrm{Ch}(\nabla)$. Define a map $K\colon CC_k(\mathcal{A}) \to$

$\Omega^\bullet(X) \otimes \mathbb{C}((u))$ of degree one by

$$K = \int_0^1 u^{-1} e^{u\beta_t} \iota_{\partial_t} \tilde{\mathrm{Ch}}\, dt. \tag{22}$$

Here $\iota_{\partial_t}$ denotes contraction with the vector field $\partial_t$ on $[0,1]$. $K$ induces maps, also denoted $K$, $K\colon CC_k(\mathcal{A}) \otimes \mathbb{C}((u)) \to \Omega^\bullet(X) \otimes \mathbb{C}((u))$ and $K\colon CC_k(\mathcal{A}) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]] \to \Omega^\bullet(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$. We have the following Proposition.

**Proposition 5.6** *The maps* $K\colon CC_\bullet(\mathcal{A}) \otimes \mathbb{C}((u)) \to \Omega^\bullet(X) \otimes \mathbb{C}((u))$ *and* $K\colon CC_\bullet(\mathcal{A}) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]] \to \Omega^\bullet(X) \otimes \mathbb{C}((u))/u\mathbb{C}[[u]]$ *defined above are chain homotopies — we have the formula*

$$e^{u\beta}\mathrm{Ch}(\nabla') - \mathrm{Ch}(\nabla) = K \circ (b + uB) - (u\mathrm{d} - u^2 c) \circ K. \tag{23}$$

**Proof.** From equation (17) we have $\tilde{\mathrm{Ch}} \circ (b + uB) = (u(\mathrm{d} + \mathrm{d}t\partial_t) - u^2(c + \mathrm{d}\beta_t + \mathrm{d}t\dot{\beta}_t) \circ \tilde{\mathrm{Ch}}$. Multiply both sides of this equation by $e^{u\beta_t}$. Then we get that

$$e^{u\beta_t}\tilde{\mathrm{Ch}} \circ (b + uB) = (u\mathrm{d} - u^2 c) \circ e^{u\beta_t}\tilde{\mathrm{Ch}} + u\mathrm{d}t\partial_t e^{u\beta_t}\tilde{\mathrm{Ch}}.$$

Multiplying both sides now by $u^{-1}$ and integrating from $t = 0$ to $t = 1$ gives the result.

**Remark 5.7** *If we look at the part of equation* (23) *which is of degree zero in $u$, then we find that we have* $\mathrm{Ch}(\nabla') - \mathrm{Ch}(\nabla) = K_0 \circ b$ *where* $\mathrm{Ch}(\nabla')$ *and* $\mathrm{Ch}(\nabla)$ *are the maps given in equation* (16)*. Here $K_0$ is the degree zero in $u$ component of the map* (22) *above so that*

$$K_0 = -\int_0^1 \beta_t \iota_{\partial_t} \tilde{\mathrm{Ch}}\, dt.$$

*This shows that the chain map* $\mathrm{Ch}_\bullet\colon CC_\bullet(\mathcal{A}) \to \Omega^\bullet(X)$ *from the Hochschild complex to the complex* (20) *induces a map on homology groups which is independent of the choice of connection* $\nabla$.

### *5.3 Proof of the Main Result*

From now on, unless we explicitly state otherwise, we denote the completed projective tensor product $\hat{\otimes}_\pi$ by $\otimes$. Let $\{U_i\}_{i\in I}$ be a good open cover of the compact manifold $X$. Let $\mathcal{A}_i$ denote the algebra $C^\infty(\mathcal{L}^1(P)|_{U_i})$ of sections of $\mathcal{L}^1(P)$ over $U_i$, $\mathcal{A}_{ij}$ denote the algebra $C^\infty(\mathcal{L}^1(P)|_{U_{ij}})$ of sections of $\mathcal{L}^1(P)$ over $U_{ij}$ and so on. We can identify the algebra $\oplus_i\mathcal{A}_i$ with the algebra

$C^\infty(\sqcup \mathcal{L}^1(P)|_{U_i}, \sqcup U_i)$ of smooth sections of the bundle $\sqcup \mathcal{L}^1(P)_{U_i}$ over the disjoint union $\sqcup U_i$. We can make similar identifications of $\oplus \mathcal{A}_{ij}$ and so on. The sequence

$$0 \to \mathcal{A} \xrightarrow{\delta} \bigoplus_i \mathcal{A}_i \xrightarrow{\delta} \bigoplus_{i,j} \mathcal{A}_{ij} \xrightarrow{\delta} \cdots \tag{24}$$

is then exact where the map $\delta$ is defined as follows. Suppose that $\underline{s} = (s_{i_1 \ldots i_p}) \in \oplus_{i_1,\ldots,i_p} \mathcal{A}_{i_1 \cdots i_p}$. Then we put

$$\delta(\underline{s})_{i_1 \cdots i_{p+1}} = s_{i_2 \cdots i_{p+1}}|_{U_{i_1 \cdots i_{p+1}}} - s_{i_1 i_3 \cdots i_{p+1}}|_{U_{i_1 \cdots i_{p+1}}} + \cdots + (-1)^{p+1} s_{i_1 \cdots i_p}|_{U_{i_1 \cdots i_{p+1}}}.$$

Another way to think of $\delta$ is the following. There are $p+1$ inclusions $\sqcup U_{i_1 \cdots i_{p+1}} \to \sqcup U_{i_1 \cdots i_p}$ which induce $p+1$ restriction homomorphisms $d_i^* \colon \oplus_{i_1,\ldots,i_p} \mathcal{A}_{i_1 \cdots i_{p+1}} \to \oplus_{i_1,\ldots,i_p} \mathcal{A}_{i_1 \cdots i_p}$. $\delta$ is then the alternating sum $\delta = \sum (-1)^i d_i^*$. To see that the sequence above is exact, suppose that $\delta(\underline{s}) = 0$. Define $\underline{t} = (t_{i_1 \cdots i_{p-1}}) \in \oplus_{i_1,\ldots,i_{p-1}} \mathcal{A}_{i_1 \cdots i_{p-1}}$ by $t_{i_1 \cdots i_{p-1}} = (-1)^p \sum_{i_p} \rho_{i_p} s_{i_1 \cdots i_p}$ for some partition of unity $\{\rho_i\}_{i \in I}$ subordinate to the open cover $\{U_i\}_{i \in I}$. For each $i_p \in I$ such that $U_{i_1 \cdots i_p} \neq \emptyset$, $(-1)^p \rho_{i_p} s_{i_1 \cdots i_p}$ defines a section of $\mathcal{L}^1(P)|_{U_{i_1 \cdots i_{p-1}}}$. It is then easy to see that $\delta(\underline{t})_{i_1 \cdots i_p} = s_{i_1 \cdots i_p}$.

This exactness argument generalises for the algebra $\mathcal{A}^{\otimes n}$. More precisely the sequence

$$0 \to \mathcal{A}^{\otimes n} \xrightarrow{\delta} (\bigoplus_i \mathcal{A}_i)^{\otimes n} \xrightarrow{\delta} (\bigoplus \mathcal{A}_{ij})^{\otimes n} \xrightarrow{\delta} \cdots \tag{25}$$

is exact. To see this, first observe that

$$(\bigoplus_i \mathcal{A}_i)^{\otimes n} = \bigoplus_{(i_1,\ldots,i_n)} \mathcal{A}_{i_1} \otimes \cdots \mathcal{A}_{i_n} = \bigoplus_{(i_1,\ldots,i_n)} C^\infty(\mathcal{L}^1(P)^{\boxtimes n}, U_{i_1} \times \cdots U_{i_n}).$$

We have similar identifications of the other tensor products $(\oplus_{i_1,\ldots,i_p} \mathcal{A}_{i_1 \cdots i_p})^{\otimes n}$; for example $(\oplus_{i,j} \mathcal{A}_{ij})^{\otimes n}$ is equal to

$$\bigoplus_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)} C^\infty(\mathcal{L}^1(P)^{\boxtimes n}, (U_{i_1} \times \cdots \times U_{i_n}) \cap (U_{j_1} \times \cdots \times U_{j_n})).$$

Recall also that $\mathcal{A}^{\otimes n} = C^\infty(\mathcal{L}^1(P)^{\boxtimes n}, X^n)$. Therefore we are in the situation above with $X$ replaced by $X^n$ and the cover $\{U_i\}_{i \in I}$ of $X$ replaced by the cover $\{U_{i_1} \times \cdots \times U_{i_n}\}_{(i_1,\ldots,i_n) \in I^n}$ of $X^n$. The same partition of unity argument used above shows that the sequence is exact. Let us describe in more detail the map $\delta$. First we need some notation. Let us write $\underline{i} = (i_1, \ldots, i_n)$ and $U_{\underline{i}} = U_{i_1} \times \cdots \times U_{i_n}$. Then $(\underline{i}_1, \ldots, \underline{i}_p) = ((i_1^1, \ldots, i_1^n), (i_2^1, \ldots, i_2^n), \ldots, (i_p^1, \ldots, i_p^n))$. The $p+1$ injections $\sqcup U_{i_1 \cdots i_{p+1}} \to \sqcup U_{i_1 \cdots i_p}$ induce $p+1$ injections $\sqcup U_{\underline{i}_1} \cap \cdots \cap U_{\underline{i}_{p+1}} \to \sqcup U_{\underline{i}_1} \cap \cdots \cap U_{\underline{i}_p}$ and hence $p+1$ restriction homomorphisms

$$d_i^* \colon \bigoplus_{\underline{i}_1,\ldots,\underline{i}_p} C^\infty(\mathcal{L}^1(P)^{\boxtimes n}, U_{\underline{i}_1} \cap \cdots \cap U_{\underline{i}_p}) \to C^\infty(\mathcal{L}^1(P)^{\boxtimes n}, U_{\underline{i}_1} \cap \cdots \cap U_{\underline{i}_{p+1}}).$$

$\delta$ is again the alternating sum $\sum (-1)^i d_i^*$. We have the following Proposition.

**Proposition 5.8** *We have exact sequences of complexes*

$$0 \to CC_\bullet(\mathcal{A}) \xrightarrow{\delta} CC_\bullet(\bigoplus_i \mathcal{A}_i) \xrightarrow{\delta} CC_\bullet(\bigoplus_{i<j} \mathcal{A}_{ij}) \xrightarrow{\delta} \cdots \tag{26}$$

$$0 \to CC_\bullet(\mathcal{A}) \otimes \frac{\mathbb{C}((u))}{u\mathbb{C}[[u]]} \xrightarrow{\delta} CC_\bullet(\bigoplus_i \mathcal{A}_i) \otimes \frac{\mathbb{C}((u))}{u\mathbb{C}[[u]]} \xrightarrow{\delta} CC_\bullet(\bigoplus_{i<j} \mathcal{A}_{ij}) \otimes \frac{\mathbb{C}((u))}{u\mathbb{C}[[u]]} \xrightarrow{\delta} \cdots \tag{27}$$

$$0 \to CC_\bullet(\mathcal{A}) \otimes \mathbb{C}((u)) \xrightarrow{\delta} CC_\bullet(\bigoplus_i \mathcal{A}_i) \otimes \mathbb{C}((u)) \xrightarrow{\delta} CC_\bullet(\bigoplus_{i<j} \mathcal{A}_{ij}) \otimes \mathbb{C}((u)) \xrightarrow{\delta} \cdots \tag{28}$$

**Proof.** To prove this we have to show that the map $\delta\colon (\oplus\mathcal{A}_{i_1\cdots i_p})^{\otimes n} \to (\oplus\mathcal{A}_{i_1\cdots i_{p+1}})^{\otimes n}$ is compatible with the maps $b'$, $b$, $\lambda$ and $B$ of Section 3 . But $b'$, $b$, $\lambda$ and $B$ commute with all the homomorphisms $d_i^*$ and hence commute with $\delta$.

This Proposition allows us to give a 'Čech-de Rham style' proof of the following theorem.

**Theorem 5.9** *Let $X$ be a compact manifold and let $P \to X$ be a principal $PU$ bundle with connection 1-form $\Theta$. Let $c = c(\Theta)$ denote the de Rham representative of the characteristic class in $H^3(X;\mathbb{Z})$ associated to $P$. Then the following results are true:*

*(1) The chain map* Ch *from the Hochschild complex $CC_\bullet(\mathcal{A})$ to the complex* (20) *is a quasi-isomorphism.*
*(2) The chain map* Ch *from the cyclic complex $CC_\bullet(\mathcal{A})\otimes\mathbb{C}((u))/u\mathbb{C}[[u]]$ for $\mathcal{A}$ to the complex $(\Omega^\bullet(X)\otimes\mathbb{C}((u))/u\mathbb{C}[[u]], ud-u^2c)$ is a quasi-isomorphism.*
*(3) The chain map $\mathrm{Ch}\colon CC_\bullet(\mathcal{A}) \otimes \mathbb{C}((u)) \to \Omega^\bullet(X) \otimes \mathbb{C}((u))$ from the complex computing periodic cyclic homology to the twisted de Rham complex $(\Omega^\bullet(X) \otimes \mathbb{C}((u)), ud - u^2c)$ is a quasi-isomorphism.*

**Proof.** We shall prove 1 and 3 only, the proof of 2 is identical and will be left to the reader. Let us first choose a good open cover $\{U_i\}_{i\in I}$ of the compact manifold $X$.

To prove 1 above, we form the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
CC_2(\oplus\mathcal{A}_i) & \xrightarrow{\delta} & CC_2(\oplus\mathcal{A}_{ij}) & \xrightarrow{-\delta} & CC_2(\oplus\mathcal{A}_{ijk}) & \xrightarrow{\delta} & \cdots \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
CC_1(\oplus\mathcal{A}_i) & \xrightarrow{\delta} & CC_1(\oplus\mathcal{A}_{ij}) & \xrightarrow{-\delta} & CC_1(\oplus\mathcal{A}_{ijk}) & \xrightarrow{\delta} & \cdots \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
CC_0(\oplus\mathcal{A}_i) & \xrightarrow{\delta} & CC_0(\oplus\mathcal{A}_{ij}) & \xrightarrow{-\delta} & CC_0(\oplus\mathcal{A}_{ijk}) & \xrightarrow{\delta} & \cdots
\end{array} \tag{29}$$

and notice that Ch defines a map of double complexes between the double complex (29) above and the following double complex.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
\Omega^2(\coprod U_i) & \xrightarrow{\delta} & \Omega^2(\coprod U_{ij}) & \xrightarrow{-\delta} & \Omega^2(\coprod U_{ijk}) & \xrightarrow{\delta} & \cdots \\
\uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
\Omega^1(\coprod U_i) & \xrightarrow{\delta} & \Omega^1(\coprod U_{ij}) & \xrightarrow{-\delta} & \Omega^1(\coprod U_{ijk}) & \xrightarrow{\delta} & \cdots \\
\uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \\
\Omega^0(\coprod U_i) & \xrightarrow{\delta} & \Omega^0(\coprod U_{ij}) & \xrightarrow{-\delta} & \Omega^0(\coprod U_{ijk}) & \xrightarrow{\delta} & \cdots
\end{array} \tag{30}$$

We now consider the two spectral sequences associated to the double complex (29). If we filter by rows then it follows from Proposition 5.8 that the $E_2$ term of the resulting spectral sequence is just the Hochschild homology $HH_\bullet(\mathcal{A})$ of $\mathcal{A}$. Now consider the spectral sequence associated to (29) obtained by filtering by columns. The map Ch of double complexes described above yields a morphism of spectral sequences. Since the cover $\{U_i\}_{i\in I}$ is good, each algebra $\oplus\mathcal{A}_{i_1\cdots i_p}$ is isomorphic to the algebra $\oplus C^\infty(U_{i_1\cdots i_p})\otimes\mathcal{L}^1$. In particular, in defining the map Ch for each algebra $\oplus\mathcal{A}_{i_1\cdots i_p}$ we may take $\nabla$ to be the trivial connection d, in which case the induced map $\mathrm{Ch}(\mathrm{d})\colon HH_\bullet(\oplus\mathcal{A}_{i_1\cdots i_p}) \to \Omega^\bullet(\coprod U_{i_1\cdots i_p})$ coincides with Connes' map (6), which we know to be an isomorphism (Section 3.2). By the homotopy invariance of the map Ch (Remark 5.7) the induced maps $\mathrm{Ch}(\nabla), \mathrm{Ch}(\mathrm{d})\colon HH_\bullet(\oplus\mathcal{A}_{i_1\cdots i_p}) \to \Omega^\bullet(\coprod U_{i_1\cdots i_p})$ also coincide. It follows then that the induced morphism of spectral sequences is an isomorphism on the $E_1$ terms and hence induces an isomorphism on the $E_\infty$ terms. This completes the proof of 1.

To prove 3, we first form the following double complex.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \\
CC_1(\bigoplus_i \mathcal{A}_i)((u)) & \xrightarrow{\delta} & CC_1(\bigoplus_{i<j} \mathcal{A}_{ij})((u)) & \xrightarrow{-\delta} & CC_1(\bigoplus_{i<j<k} \mathcal{A}_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \\
CC_0(\bigoplus_i \mathcal{A}_i)((u)) & \xrightarrow{\delta} & CC_0(\bigoplus_{i<j} \mathcal{A}_{ij})((u)) & \xrightarrow{-\delta} & CC_0(\bigoplus_{i<j<k} \mathcal{A}_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \\
CC_{-1}(\bigoplus_i \mathcal{A}_i)((u)) & \xrightarrow{\delta} & CC_{-1}(\bigoplus_{i<j} \mathcal{A}_{ij})((u)) & \xrightarrow{-\delta} & CC_{-1}(\bigoplus_{i<j<k} \mathcal{A}_{ijk})((u)) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \uparrow{\scriptstyle b+uB} & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{31}$$

where $CC_k(\mathcal{A})((u))$ denotes the elements of total degree $k$ in $CC_k(\mathcal{A})\otimes\mathbb{C}((u))$ as in § 4.3. Associated to this double complex are two spectral sequences obtained by filtering by rows or by columns. If we filter by rows then it follows from Proposition 5.8 above that the $E_2$ term of the resulting spectral sequence is the periodic cyclic homology of the algebra $\mathcal{A}$. On the other hand, the map Ch induces a map of bicomplexes from the bicomplex (31) to the bicomplex (14) and hence a morphism of the spectral sequences associated to these bicomplexes by filtering with respect to columns. As in the proof of 1, it follows from Proposition 5.6 and Remark 3.6 that this morphism of spectral sequences is an isomorphism on the $E_1$ terms. This proves the 3.

Recall that through the adjoint action the projective unitary group $PU$ acts on the Schatten ideals $\mathcal{L}^p$, $p \geq 1$. Therefore for any $p \geq 1$ we can form the associated bundle $\mathcal{L}^p(P)$ and the Fréchet algebra of smooth sections $C^\infty(\mathcal{L}^p(P))$. The $K$-theory of the algebra $C^\infty(\mathcal{L}^p(P))$ is the twisted $K$-theory $K_i(C(\mathcal{K}(P)))$. We can, with very little extra effort, compute the periodic cyclic homology of the algebra $C^\infty(\mathcal{L}^p(P))$ using Theorem 5.9. The inclusion $\mathcal{L}^1 \hookrightarrow \mathcal{L}^p$ induces an inclusion $\mathcal{L}^1(P) \hookrightarrow \mathcal{L}^p(P)$. Associated to the two algebras $C^\infty(\mathcal{L}^1(P))$ and $C^\infty(\mathcal{L}^p(P))$ and the good cover $\{U_i\}_{i\in I}$ of $X$ we have double complexes of the form (31). The inclusion $\mathcal{L}^1(P) \hookrightarrow \mathcal{L}^p(P)$ induces a morphism of these double complexes. Recall that the inclusion $\mathcal{L}^1 \hookrightarrow \mathcal{L}^p$ induces an isomorphism on periodic cyclic homology. More generally we have ([13] Corollary 17.2) if $U$ is an open subset of $X$ then the inclusion $C^\infty(U)\otimes\mathcal{L}^1 \hookrightarrow C^\infty(U)\otimes\mathcal{L}^p$ induces an isomorphism on periodic cyclic homology. Consider the double complexes (31) associated to the algebras $C^\infty(\mathcal{L}^1(P))$ and $C^\infty(\mathcal{L}^p(P))$. If we filter the respective total complexes by columns then the morphism $C^\infty(\mathcal{L}^1(P)) \hookrightarrow C^\infty(\mathcal{L}^p(P))$ in-

duces an isomorphism on the $E_1$ terms of the resulting spectral sequences. It follows then that $HP_i(C^\infty(\mathcal{L}^1(P)))$ and $HP_i(C^\infty(\mathcal{L}^p(P)))$ are isomorphic.

**Corollary 5.10** *For $p \geq 1$ the inclusion $C^\infty(\mathcal{L}^1(P)) \hookrightarrow C^\infty(\mathcal{L}^p(P))$ induces an isomorphism on periodic cyclic homology.*

## 6 The Twisted Chern Character

In this section we relate the Connes-Chern character $\mathrm{ch}\colon K_\bullet(\mathcal{A}) \to HP_\bullet(\mathcal{A})$ to the twisted Chern character developed in earlier work [6]. Notice that we can view ch as a homomorphism $\mathrm{ch}_P\colon K^\bullet(X,P) \to H^\bullet(X,c(P))$ once we notice the identifications $K_\bullet(\mathcal{A}) \cong K^\bullet(X,P)$ and $HP_\bullet(\mathcal{A}) \cong H^\bullet(X,c(P))$ where the first isomorphism is established in [34] and the second isomorphism is Theorem 5.9.

We first observe that the Connes-Chern character ch, or alternatively the homomorphism $\mathrm{ch}_P\colon K^\bullet(X,P) \to H^\bullet(X,c(P))$, becomes an isomorphism after tensoring with the complex numbers. Recall that in general one has no assurances that the Connes-Chern map $\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$ will be an isomorphism after tensoring with $\mathbb{C}$ for arbitrary algebras $\mathcal{A}$ — for example $K_i(C(X)) = K^i(X)$ but $HP_i(C(X)) = 0$ where $C(X)$ is the algebra of continuous complex valued functions on a compact space $X$. However, in our situation, we can prove the following.

**Proposition 6.1** *For the algebra $\mathcal{A} = C^\infty(\mathcal{L}^1(P))$, the Connes-Chern character $\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$ becomes an isomorphism after tensoring with $\mathbb{C}$, for $i = 0, 1$.*

**Proof.** We prove this Proposition by making repeated use of the Mayer-Vietoris exact sequence (note first that the construction of the Mayer-Vietoris sequence given in [3], pages 219–220, goes across to the periodic cyclic theory $HP$). First of all choose a cover $\{B_\alpha\}_{\alpha\in A}$ of $X$ by closed, contractible sets $B_\alpha$ (this can be done using a minor modification of the usual construction of a good open cover by geodesically convex balls). Consider the Mayer-Vietoris sequence for the pair of closed sets $\{B_\alpha, B_\beta\}$. Let us write $\mathcal{A}_{B_\alpha}$ and $\mathcal{A}_{B_{\alpha\beta}}$ for the algebras $C^\infty(\mathcal{L}^1(P)|_{B_\alpha})$ and $C^\infty(\mathcal{L}^1(P)|_{B_{\alpha\beta}})$ respectively, where $B_{\alpha\beta} = B_\alpha \cap B_\beta$. Then the Mayer-Vietoris exact sequence for $\mathcal{A}|_{B_\alpha\cup B_\beta}$ in $K$-theory is

$$\to K_i(\mathcal{A}|_{B_\alpha\cup B_\beta}) \to K_i(\mathcal{A}|_{B_\alpha})\oplus K_i(\mathcal{A}|_{B_\beta}) \to K_i(\mathcal{A}|_{B_{\alpha\beta}}) \to K_{i+1}(\mathcal{A}|_{B_\alpha\cup B_\beta}) \to \tag{32}$$

and there is a similar exact sequence for the periodic cyclic theory. From results of Nistor [30], we see that the Connes-Chern map ch furnishes us with a morphism from the exact sequence (32) to the corresponding sequence for $HP$. We know that $\mathrm{ch}\colon K_i(\mathcal{A}|_{B_\alpha}) \to HP_i(\mathcal{A}|_{B_\alpha})$ and $\mathrm{ch}\colon K_i(\mathcal{A}|_{B_{\alpha\beta}}) \to HP_i(\mathcal{A}|_{B_{\alpha\beta}})$ become isomorphisms after tensoring with $\mathbb{C}$ for all $\alpha$ and $\beta$. It follows therefore by the 5-lemma that $\mathrm{ch}\colon K_i(\mathcal{A}|_{B_\alpha \cup B_\beta}) \to HP_i(\mathcal{A}|_{B_\alpha \cup B_\beta})$ is an isomorphism after tensoring with $\mathbb{C}$. By induction we see that $\mathrm{ch}\colon K_i(\mathcal{A}) \otimes \mathbb{C} \to HP_i(\mathcal{A}) \otimes \mathbb{C}$ is an isomorphism.

Upon identifying $K_i(\mathcal{A})$ with $K^i(X,P)$ (see [6]) and $HP_i(\mathcal{A})$ with $H^i(X,c(P))$ (see Theorem 1.1), we denote the Connes-Chern character $\mathrm{ch}\colon K_i(\mathcal{A}) \to HP_i(\mathcal{A})$ by $\widetilde{\mathrm{ch}}_P\colon K^i(X,P) \to H^i(X,c(P))$. The following Proposition is easy to establish.

**Proposition 6.2** *The Connes-Chern character* $\widetilde{\mathrm{ch}}_P\colon K^i(X,P) \to H^i(X,c(P))$ *satisfies the following properties:*

*(1) If the bundle $P$ is trivial (say $P$ is trivialised by a section $s$) then $\widetilde{\mathrm{ch}}_P$ reduces to the ordinary Chern character $\mathrm{ch}\colon K^i(X) \to H^i(X)$ in the sense that we have the commutative diagram*

$$\begin{tikzcd} K^i(X,P) \arrow[r, "\widetilde{\mathrm{ch}}_P"] \arrow[d, "\cong"'] & H^i(X,c(P)) \arrow[d, "e^{us^*f}"] \\ K^i(X) \arrow[r, "\mathrm{ch}"'] & H^i(X) \end{tikzcd}$$

*(2) $\widetilde{\mathrm{ch}}_P\colon K^i(X,P) \to H^i(X,c(P))$ is natural with respect to maps $h\colon Y \to X$ in the sense that we have the following commutative diagram*

$$\begin{tikzcd} K^i(X,P) \arrow[r, "\widetilde{\mathrm{ch}}_P"] \arrow[d, "h^*"'] & H^i(X,c(P)) \arrow[d, "h^*"] \\ K^i(Y,h^*P) \arrow[r, "\widetilde{\mathrm{ch}}_{h^*P}"] & H^i(Y,h^*c(P)). \end{tikzcd}$$

In (1) above, the de Rham cocycle $c(P)$ is trivialised by $s$ since we have $c(P) = \mathrm{d}s^*f$, where $f$ is the 2-form (11).

In [6], a geometric, twisted Chern character was defined,

$$\mathrm{ch}_P : K^\bullet(X,P) \to H^\bullet(X,c(P)), \tag{33}$$

using connections and curvature as well as a curving for the principal $PU$ bundle $P$, which has the same functorial properties as the Connes-Chern character, namely that is natural with respect to maps and that it reduces to the ordinary

Chern character when $P$ is the trivial $PU$ bundle. The twisted Chern character also satisfies other functorial properties, such as being compatible with the $K^0(X)$ module structure of $K^\bullet(X,P)$ and the $H^{even}(X)$ module structure of $H^\bullet(X,c(P))$, cf. [27]. The analogous properties for the Connes-Chern character can be deduced from the following basic result that identifies the two homomorphisms.

**Proposition 6.3** *For the algebra $\mathcal{A} = C^\infty(\mathcal{L}^1(P))$, the Connes-Chern isomorphism $\mathrm{ch}\colon K_i(\mathcal{A})\otimes\mathbb{C} \to HP_i(\mathcal{A})\otimes\mathbb{C}$ coincides with the twisted Chern character ismorphism $\mathrm{ch}_P\colon K^i(X,P)\otimes\mathbb{C} \to H^i(X,c(P))\otimes\mathbb{C}$ for $i=0,1$.*

**Proof.** We follow the strategy of Proposition 6.1. Let $\{B_\alpha\}_{\alpha\in A}$ be a (finite) cover of the compact manifold $X$ by closed balls $B_\alpha$ such that $\mathcal{L}^1(P)$ is trivialised over $B_\alpha$. Consider the Mayer-Vietoris exact sequences in $K$-theory and periodic cyclic homology for the algebra $\mathcal{A}|_{B_\alpha}$. A choice of local section $s_\alpha : B_\alpha \to P\Big|_{B_\alpha} = P_\alpha$, trivializes $\mathcal{A}|_{B_\alpha} \cong C^\infty(B_\alpha)\otimes\mathcal{L}^1$. Using this local section, we will have the commutative diagram

$$
\begin{array}{ccccccc}
K_\bullet(C^\infty(B_\alpha)\otimes\mathcal{L}^1) & \xrightarrow{\cong} & K_\bullet(C^\infty(B_\alpha)) & \xrightarrow{\cong} & K^\bullet(B_\alpha) & \xrightarrow{\cong} & K^\bullet(B_\alpha,P_\alpha) \\
\Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}_{P_\alpha}} \\
HP_\bullet(C^\infty(B_\alpha)\otimes\mathcal{L}^1) & \xrightarrow{\cong} & HP_\bullet(C^\infty(B_\alpha)) & \xrightarrow{\cong} & H^\bullet(B_\alpha) & \xrightarrow{\cong} & H^\bullet(B_\alpha,c(P_\alpha))
\end{array}
\tag{34}
$$

where the horizontal arrows are all isomorphisms, and the vertical arrows are isomorphisms after tensoring with $\mathbb{C}$. The first and second vertical arrows starting from the left are the Connes-Chern isomorphism. The third vertical arrow is the usual Chern character, which is well known can be identified with the Connes-Chern character on smooth functions on a manifold. The last isomorphism uses the local section again, where $s_\alpha^* f = f_\alpha$ and $s_\alpha^* c = c_\alpha = df_\alpha$, and the last vertical arrow is the twisted Chern character. The first upper horizontal isomorphism is essentially Morita equivalence. More precisely, one uses fact that $C^\infty(B_\alpha)\otimes\mathcal{L}^1$ is a smooth subalgebra of $C(B_\alpha)\otimes\mathcal{K}$, so that the inclusion map induces an isomorphism in $K$-theory. Then one uses the invariance of $K$-theory upon tensoring with the compact operators $\mathcal{K}$. The second upper horizontal isomorphism is the Serre-Swan isomorphism, the third is the fact the the local section trivializes the $PU$ bundle over $B_\alpha$ and we use Morita invariance again. The first lower horizontal isomorphism is Morita equivalence, the second is the Connes-Hochschild-Kostant-Rosenberg isomorphism, the third isomorphism is the fact that the twisted deRham differential is obtained by conjugating the standard one by $e^{uf_\alpha}$. In summary, a local section $s_\alpha : B_\alpha \to P\Big|_{B_\alpha} = P_\alpha$ gives rise to the commutative diagram,

$$\begin{array}{ccc} K_\bullet(\mathcal{A}|_{B_\alpha}) & \xrightarrow{\cong} & K^\bullet(B_\alpha, P_\alpha) \\ \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}_{P_\alpha}} \\ HP_\bullet(\mathcal{A}|_{B_\alpha}) & \xrightarrow{\cong} & H^\bullet(B_\alpha, c(P_\alpha)) \end{array}$$

where the upper horizontal arrow is the isomorphism in [6] and is the result of the composition of the upper horizontal isomorphisms in equation (34): the lower horizontal arrow is the generalized Connes-Hochschild-Kostant-Rosenberg isomorphism that is established in this paper, and is the result of the composition of the lower horizontal isomorphisms in equation (34), We next use the Mayer-Vietoris sequence, the 5-lemma and induction as in Proposition 6.1 to establish the proposition.

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